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THIS IS VOLUME 3 OF A THREE-VOLUME EXPERIMENTAL EDITION CONTAINING A SEQUENCE OF ENRICHED MATERIALS FOR SEVENTH-GRADE MATHEMATICS. THESE MATERIALS ARE DESIGNED TO BE USED FOR A PROGRAM OF INDIVIDUALIZED INSTRUCTION FOR THE ACCELERATED STUDENT OR FOR CLASSROOM PRESENTATION BY THE TEACHER. THE PRESENTATION OF THE MATERIAL IS SUCH AS TO REFLECT CHANGES IN CONTENT, TECHNIQUE, APPROACH AND EMPHASIS. INSTRUCTIONAL UNITS ON A NUMBER OF SEQUENTIALLY RELATED TOPICS ARE DESIGNED TO INCORPORATE MODERN TERMINOLOGY WITH THE TRADITIONAL TOPICS AND TO INTRODUCE NEW CONCEPTS AS APPROPRIATE. THIS VOLUME INCLUDES MATERIALS FOR (1) TRANSFORMATIONS AND ORIENTATIONS OF THE PLANE, (2) SEGMENTS, ANGLES, AND ISOMETRIES, (3) ELEMENTARY NUMBER THEORY, (4) THE RATIONAL NUMBERS, (5) MASS POINTS, (6) SOME APPLICATIONS OF THE RATIONAL NUMBERS, AND (7) INCIDENCE GEOMETRY. (RP)

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MATHEMATICS I

(Experimental Edition)

Volume 3

Secondary School Mathematics Curriculum Improvement Study

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Volume 3

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CHAPTER 9

TRANSFORMATION OF THE PLANE AND ORIENTATIONS IN THE PLANE

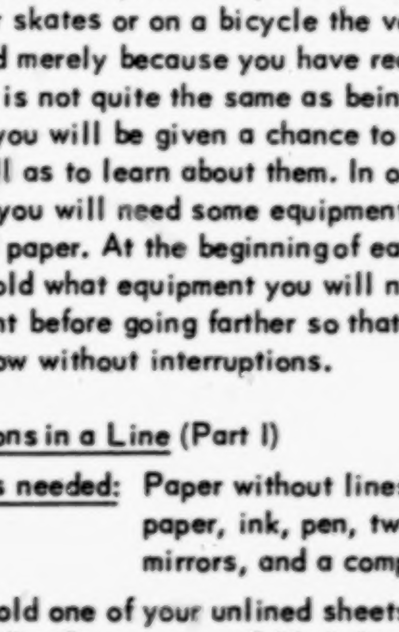
9.1 Knowing How and Doing

Have you ever read a book on how to roller skate or ride a bicycle? Do you think you could have done well on roller skates or on a bicycle the very first time you tried merely because you have read the book? Knowing how is not quite the same as being able. In this chapter you will be given a chance to do many things as well as to learn about them. In order to do these things you will need some equipment in addition to pencil and paper. At the beginning of each section you will be told what equipment you will need. Obtain this equipment before going farther so that you can read and follow without interruptions.

9.2 Reflections in a Line (Part I)

Materials needed: Paper without lines, tracing paper, ink, pen, two rectangular mirrors, and a compass.

Activity 1: Fold one of your unlined sheets of paper down the middle. Open up your folded sheet and put one drop of ink in the crease you made, and one drop of ink about an inch away from the crease.



Close the paper and spread the ink about, keeping the ink *within* the folded paper. Now open up your paper. Look at the ink spots on both paper halves. How do the ink spots compare in size and shape? Now fold one half back and replace it by one of your mirrors in an upright position so that the edge of the mirror fits into the crease. How do the images you see in the mirror compare with the ink spots you folded back?

Put 2 more ink drops on one half of your paper and repeat the steps of the preceding paragraph. Compare the distance between any 2 ink spots on one paper half with the distance for the corresponding 2 ink spots on the other. Are they the same? What generalization seems to hold for the two paper halves? Let us call the ink spot figure on one paper half the reflection in the crease of the other ink spot figure.

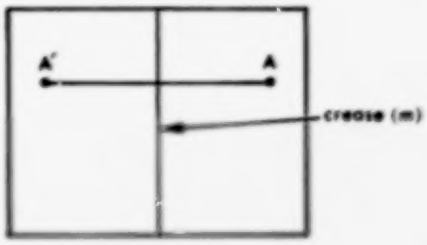
After the ink dries, use your tracing paper to trace around one of your ink spot figures. What must you do to your tracing paper to get a picture of the reflection of the figure you traced?

In previous chapters we have learned that a mapping makes assignments. For example, the successor mapping, S, assigns to each integer the next larger integer.

$$S: n \longrightarrow n + 1$$

Reflection in a line is also a mapping since it assigns points to points on a plane. Restricting ourselves to a fixed plane, a reflection with respect to a fixed line assigns to each point its mirror image or reflection in the given line. In this section we shall study properties of reflection mappings.

Activity 2: Fold one of your unlined sheets of paper down the middle. Open up your folded sheet and place a heavy dot off the crease line, label the dot "A".



Try to guess where its reflection in the crease will be. Fold along the crease with the dot inside. You should be able to see the dot through the paper. Use a ball-point or pencil to go heavily over the dot from the wrong side. Open up your paper. You should now be able to see a mark for the true image of the dot. How good was your guess? Call the actual reflection of A in m, "A'". Place another dot, B, and guess where its reflection in m ought to be. Now find the image of B under the reflection in m just as you found A'. Call the image of B, B'.

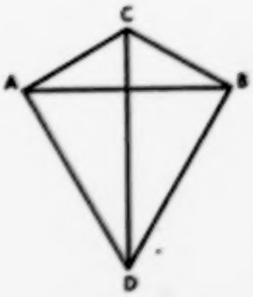
Draw a line between A and B, A' and B'. Using an opening of your compass, check to see whether the length of segment $\overline{AB}$ is the same as the length of segment $\overline{A'B'}$.

Place another point on the same half, call it "C", and try to guess where its reflection in m, C', is. Check by folding on m. Compare the lengths of $\overline{AC}$ with $\overline{A'C'}$ and of $\overline{BC}$ with $\overline{B'C'}$. How do your measurements support your generalization for Activity 1?

Join A to A' and mark the point where the line drawn crosses m, "A_1" (Read: "A one"). How do the lengths of $\overline{AA_1}$ and $\overline{A'A_1}$ compare? Join B to B', C to C′ crossing m in B_1 and C_1, respectively. How do $\overline{BB_1}$ and $\overline{B'B_1}$ compare in length? $\overline{CC_1}$ and $\overline{C'C_1}$? What generalization might you make from these observations?

The mapping with respect to a fixed line, m, that takes every point into its mirror image (such as A into A′), is called a reflection in m. You noticed above that the length of $\overline{AB}$ was the same as the length of $\overline{A'B'}$, the length of $\overline{AC}$ was the same as the length of $\overline{A'C'}$, and the length of $\overline{BC}$ was the same as the length of $\overline{B'C'}$. The mapping which assigned A to A′, B to B′, and C to C′ was such that the distance between any two points of its domain was the same as the distance between the images of these points in the range. A mapping like this, which preserves distances, is called an isometry ("iso" means equal, "metry" means measure). Do you think that every reflection is an isometry? Is every isometry a reflection?

The entire picture on the full sheet is said to be symmetric with respect to m, and m is called the line of symmetry for this full picture. What is the line of symmetry for this kite figure?

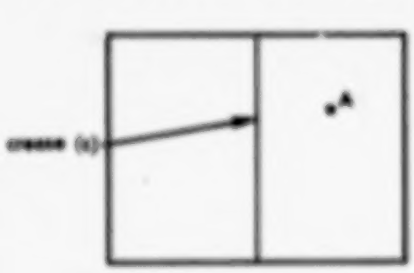


How many lines of symmetry does a rectangle have? a square?

Returning to our sheet, join A_1 to B and B′. Compare A_1B with A_1B'? Join A_1 to C and C′. Compare A_1C and A_1C'. Join any other point, P, on the crease m to A and A′, C and C′. What seems to be true about the distances of any point on m to a point and its reflection?

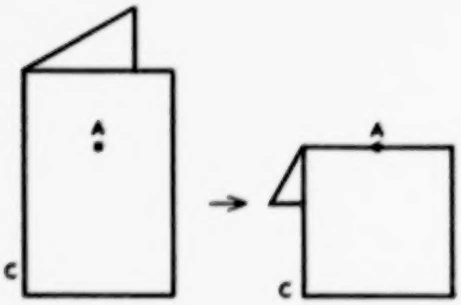
Your observations should lead you to believe that a line reflection is an isometry, and that a figure together with its reflection is symmetric with respect to the line of reflection.

Activity 3: Fold one of your unlined sheets. Open up and put a dot on one side of the crease, label it "A".



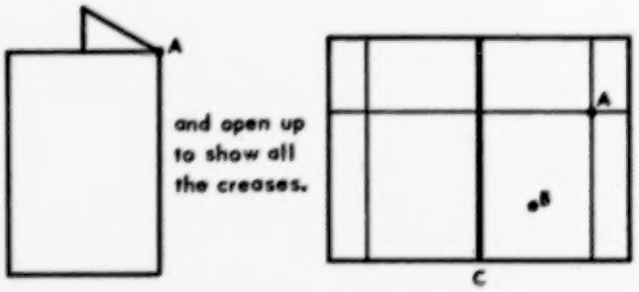
Simply by folding this paper, try to locate the reflection of A on C. Do not read further without first trying. Some hints are:

1. Fold back along the crease, and then fold back at A as shown in this figure.



Can you finish now?

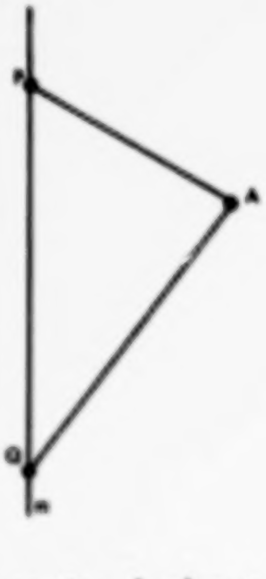
2. Fold back once again at A.



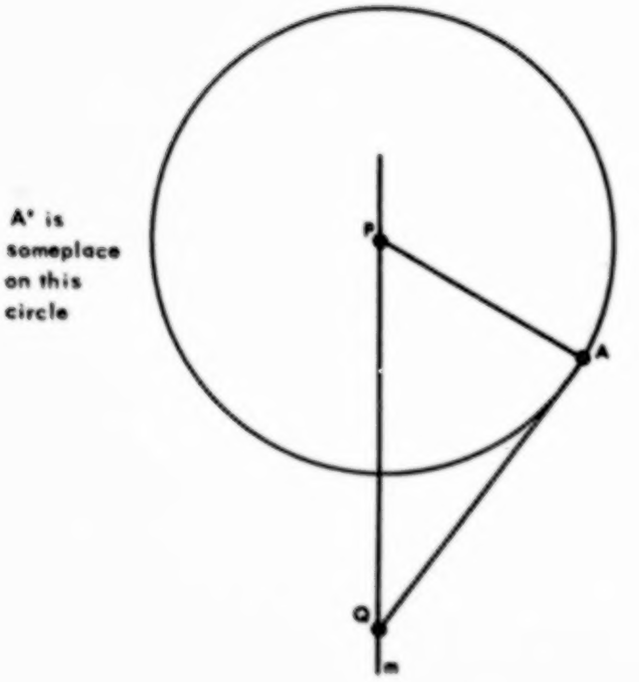
Where is A′? Find B′ the same way.

Activity 4: We shall now see how to obtain the reflection of a point in a line without folding. First try to figure out a way yourself. There are many ways of doing it. You will probably need your compass.

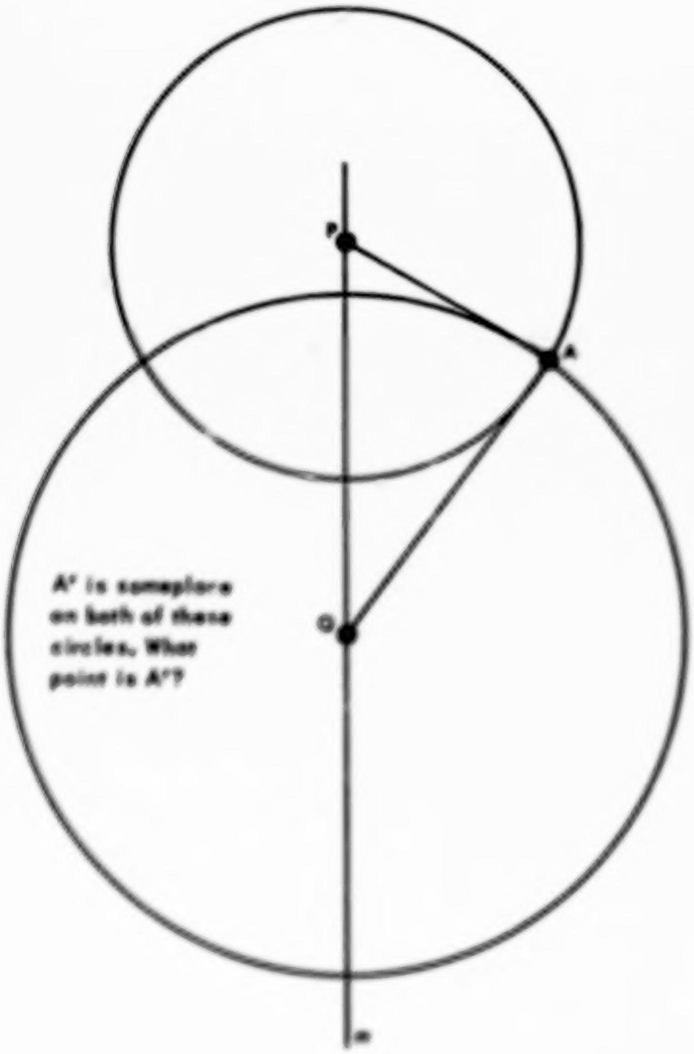
One method of finding the reflection of a point A in m is to think of the kite figure. Find 2 points in m, call them P and Q, and think of PAQ as half a kite figure.



Our previous observations lead us to believe that A′, the image of A, is just as far from P as A is from P, and that A′ is just as far from Q as A is from Q. If we draw a circle with P as center and a radius of length PA, then A′ must be someplace on this circle.

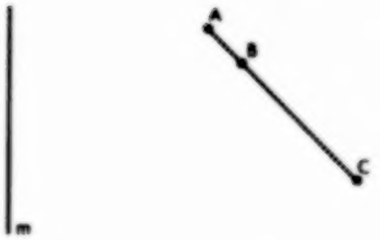


A' must also be on a circle with center Q and radius QA.



Join A' to P and Q to complete the kite figure.

Using this method of obtaining reflections, find the reflections of points A, B, C if A, B, C are on the same line with B between A and C.

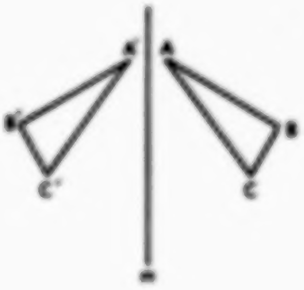


Are the image points A', B', C' also on a line? Is B' between A' and C'? What generalizations are suggested by your observations? Suppose B is taken as the midpoint of $\overline{AC}$, what is your guess about B'? Check your guess with a compass.

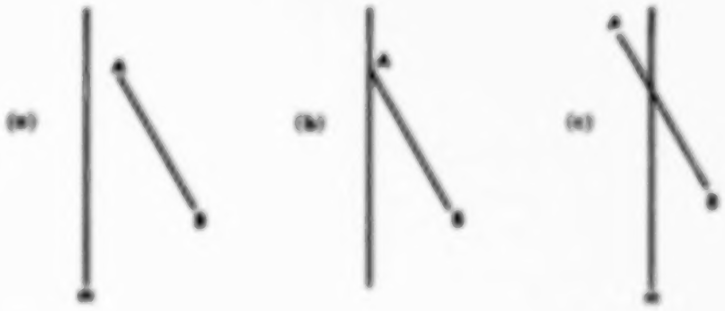
Your observations should have suggested to you that a reflection maps collinear points into collinear points preserving betweeness. That is, if P, Q, R are points on the same line, ℓ, then their images P', Q', R' are on the same line ℓ'. If Q is between P and R, then Q' is between P' and R'. In fact, the midpoint of a segment is mapped into the midpoint of the image of this segment.

9.3 Exercises

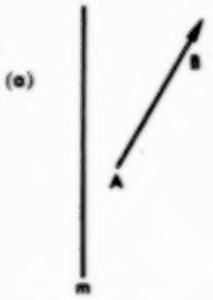
1. Which points in a plane are their own images under a line reflection?
2. If you hold a pencil in your right hand, what hand does it look like in the mirror?
3. If you spin a top clockwise, what does it seem to be doing in the mirror?
4. If points A', B', C' are the images of points A, B, C under a reflection in m, what are the images of A', B', C' under this reflection?

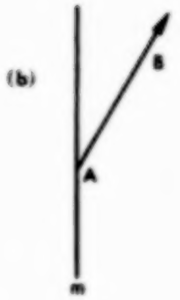


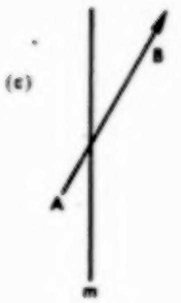
5. Draw the reflection in m of line segment $\overline{AB}$.



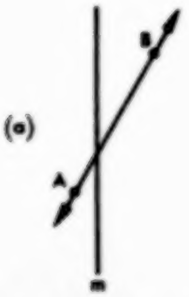
6. Draw the reflection in m of ray $\overline{AB}$.

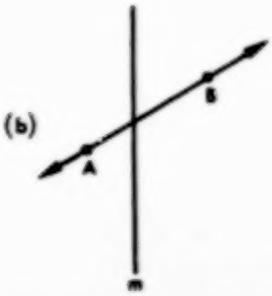




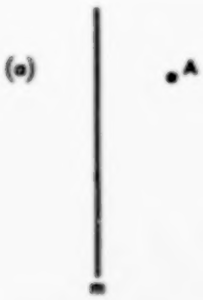


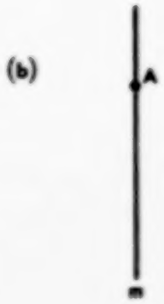
7. Draw the reflection in m of line AB.





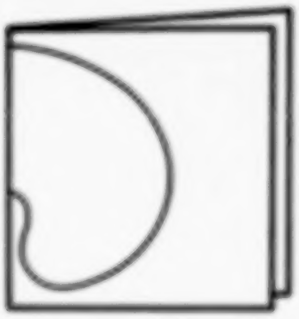
8. Find all lines through A that are identical with their reflections in m:





9. Do exercise 8 by creasing a paper on which m and A are shown, if you did not use this method in Exercise 8.

10. Fold a sheet of paper down the middle and draw some picture as shown here. Cut along the line you drew and open up. What do you notice?

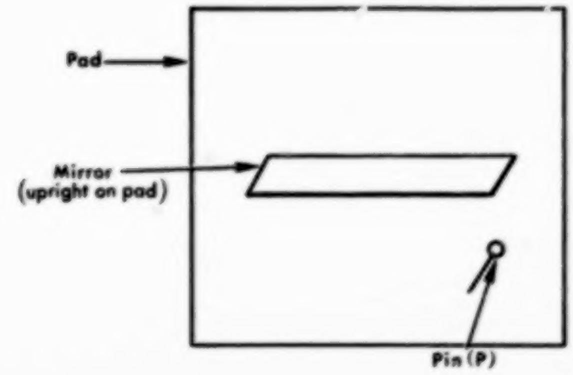


11. Which printed capital letters frequently have a line of symmetry? Will the reflection of these letters in any line be the same letters?

12. Try writing your name so that it reads right in a mirror.

13. Place a sheet of carbon paper under a sheet of paper so that the carbon faces the back side of your paper. Write your name. Look at the back side of your paper in a mirror. What do you see?

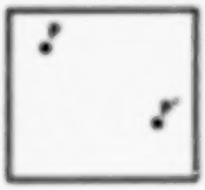
14. For this exercise you will need a pad, 2 pins, and a mirror about $\frac{1}{2}$'' wide and at least 6'' long. If you cannot get a mirror of this size, try to improvise.



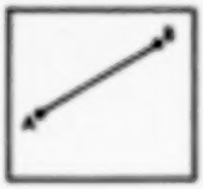
Secure the mirror in an upright position on the pad. (Brace it with a book, or fasten it with pins, scotch tape, or adhesive tape.) Stick a pin upright into the pad about 2'' in front of the mirror. Place your eye close to the pad so that you can see the image of the lower part of the pin, P, in the mirror. Try to place the other pin, P′, so that it will always line up with the image of P you see in the mirror no matter how you change your line of vision. Where is P′ in relation to P? Your pin, P′, should be located at the reflection of P in the mirror. P′ is now the image of P under a reflection in the mirror. This close analogy between a reflection mapping and reflections in a real mirror is the reason for using the words ''reflection'' and ''image''.

15. By folding your paper, find the line m, for a reflection that will map

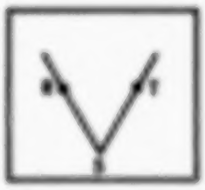
(a) P onto P′



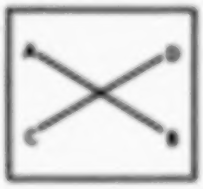
(b) $\overline{AB}$ onto itself



(c) $\overline{SR}$ onto $\overline{ST}$



(d) Line AB onto line CD. (There are 2 lines m)

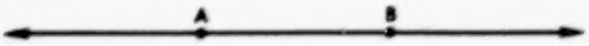


(e) In each of the above exercises what can you say about the crease?

9.4 Lines, Rays and Segments

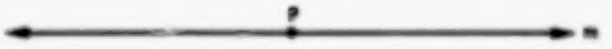
Although we picture a line as a taut string, as the edge of a molding, as a mark on the blackboard or paper, we must recognize that these things are quite inaccurate as representations of a line. For example, a string may sag or have a "belly". A string has thickness. A string does not go on and on in both directions endlessly. However, a line has no "belly", no thickness, and does go on endlessly in both directions. But how can we do any better? A line is an idea (like a number) while a physical representation is a thing (like a numeral) used to denote the idea. The marks we call "lines", only represent lines yet we still continue to refer to the marks as lines because we are not really concerned about the marks but about the ideas the marks represent.

If "A" and "B" name two points of a line then "AB" names the line containing A and B. We assume that there is only one line (our lines are always straight) that contains two different points. AB and BA are the same line.



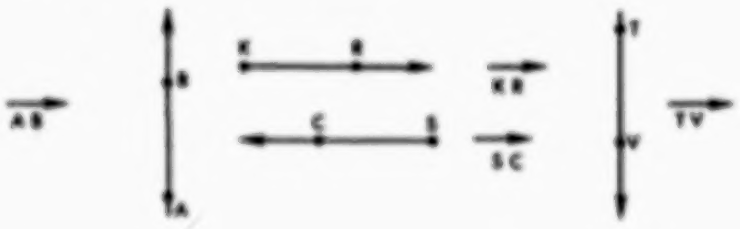
We often place arrow heads at the ends of our marks to remind us that the lines are endless in both directions. Sometimes, we place a letter on the mark and refer to the line by the letter.

Consider a line m and a point P in this line:

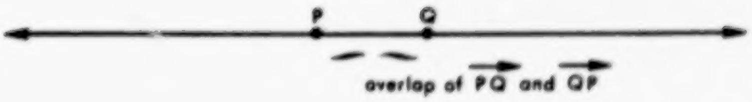


The set of points in line m to the right of P, together with P, is a ray. The set of points in m to the left of P together with P is also a ray. Point P is called the endpoint of both rays. Any point P in a line together with all the points of the line that are on the same side of P, constitute a ray.

We often name a ray by two capital letters. The left letter names the endpoint of the ray and right letter names any other point of the ray. An arrow pointing to the right is placed over both letters.



If P and Q are two points on line m, $\overrightarrow{PQ}$ and $\overrightarrow{QP}$ are different rays. They overlap on a set of points containing P, Q and all the points between P and Q.

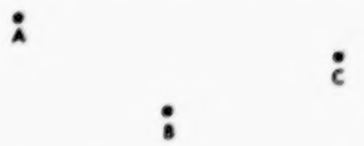


overlap of $\overrightarrow{PQ}$ and $\overrightarrow{QP}$

The overlap of PQ and QP is the segment $\overline{PQ}$ (or $\overline{QP}$).

9.5 Exercises

1. Let A, B, C, be any 3 points that are not on the same line (non-collinear points).



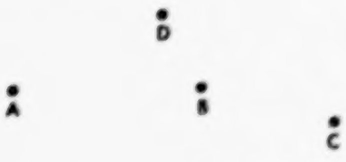
Draw all the lines you can, each containing two of these points.

(a) How many lines did you get?

(b) Name the lines.

(c) Name each of these lines another way using the same letters.

2. Let A, B, C, D be any 4 points, no three of which are collinear.



Draw all the lines you can each containing two points.

(a) How many did you get?

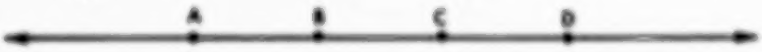
(b) Do the same thing for 5 points, no 3 of which, are collinear. Fill in the table below and try to discover a pattern that you feel should continue.

(c)

Number of Points	2	3	4	5	6
Number of Lines					

(d) Try to give an argument to support your generalization.

3.



(a) Name the line shown in as many ways as you can using the names of the given points. There are 12 possible ways.

(b) Name all the different rays you can find in the figure. Note $\overrightarrow{AB}$, $\overrightarrow{AC}$, $\overrightarrow{AD}$ are all the same ray.

(c) How many different rays did you find?

(d) Fill in the table:

Number of Points on a Line	1	2	3	4	5
Number of Rays					

(e) Try to discover a pattern that you feel ought to continue.

(f) Try to give an argument to support your generalization.

(g) Name all the segments formed by points A, B, C, D.

(h) How many different segments did you get?

(i) Fill in the table:

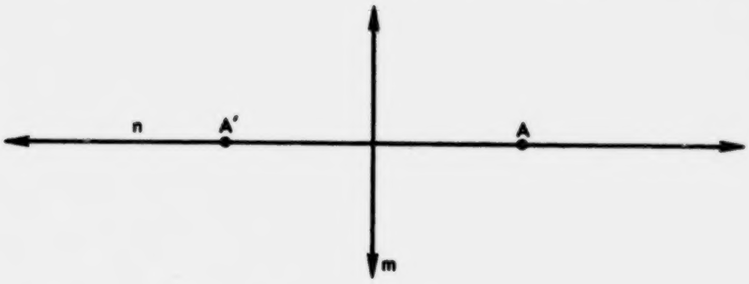
Number of Points on a Line	2	3	4	5	6
Number of Segments					

(j) Try to discover a pattern that you feel ought to continue.

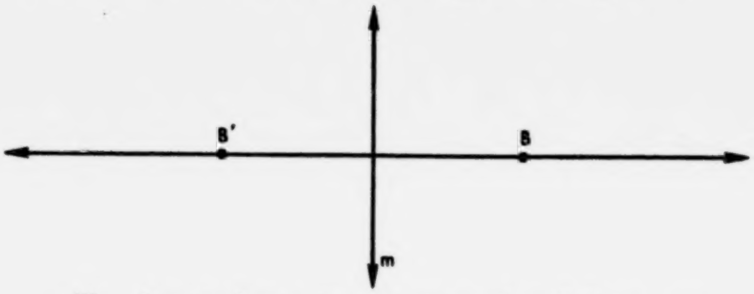
(k) Try to give an argument to support your generalization.

9.6 Perpendicular Lines

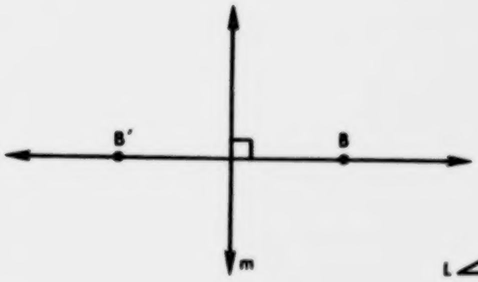
In one of the exercises you were asked to find a line, n, through A that is its own reflection in m. Your line should look like the one in the figure. Whenever we have two lines such that either is its own reflection in the other, we say that these lines are perpendicular to each other. We use the symbol "$\perp$" for "perpendicular" or "is perpendicular to". For the figure above, we have $m \perp n$ and $n \perp m$.

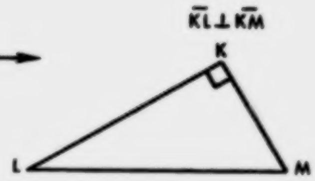


If B and B' are two points, each the reflection of the other in line m, then $BB' \perp m$, and $m \perp BB'$.



We often indicate in a drawing that 2 lines are perpendicular by a little square where the lines cross.



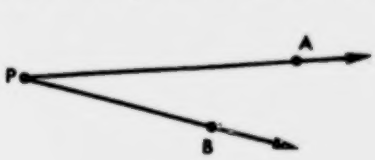


Line segments which are in perpendicular lines are said to be perpendicular. Rays which are in perpendicular lines are said to be perpendicular. In fact, any combination of line, ray and segment may be perpendicular if they are in perpendicular lines. We continue to use "$\perp$" for any such perpendicularity.

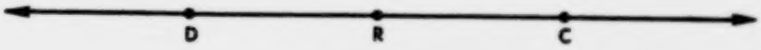
9.7 Rays Having The Same Endpoint

In this section we shall be dealing with rays that have a common endpoint.

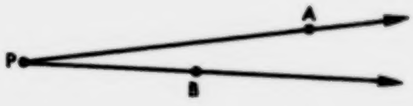
$\overrightarrow{PA}$ and $\overrightarrow{PB}$ are rays with the same endpoint, P.



If two rays with the same endpoint constitute a line, they are called opposite rays. The rays $\overrightarrow{RC}$ and $\overrightarrow{RD}$ are opposite rays.

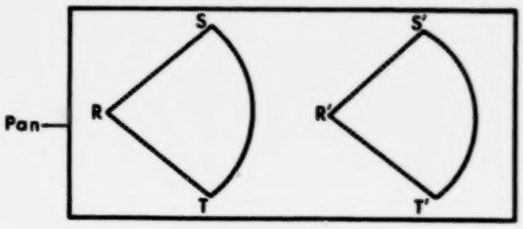


Some rays with the same endpoint have directions that are not very different. These rays have a small spread or a small opening. For instance, the rays in this figure seem to be close together.



If we were given two such pairs of rays with small spread or opening, how could we compare the openings? How could we tell which pair of rays have a greater spread? To see when such information would be handy, consider the following situation.

Mom makes delicious pies of uniform thickness. She is very skillful at cutting sections from the center. When you get home one day you see these two pieces in a pan.

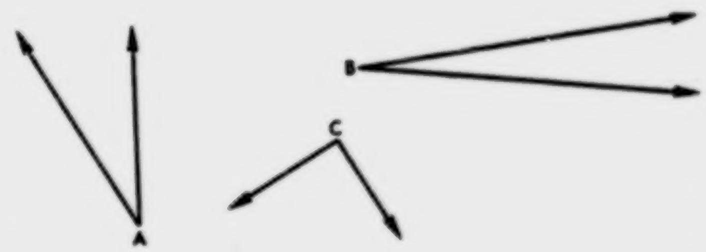


Segments $\overline{RS}$, $\overline{RT}$, $\overline{R'S'}$, and $\overline{R'T'}$ are all the same length.

Which one would you select if you want the larger piece? You may want to use your compass to help you decide. How might you use it? Think about this question a moment before reading on.

If you thought of comparing the distance from S to T with the distance from S′ to T′, then you have anticipated the text. These measurements were intended to be identical, but your eyes probably made you feel that the left piece is the larger.

Using this tasty example as a clue, how could you decide which pair of rays have the greatest spread?

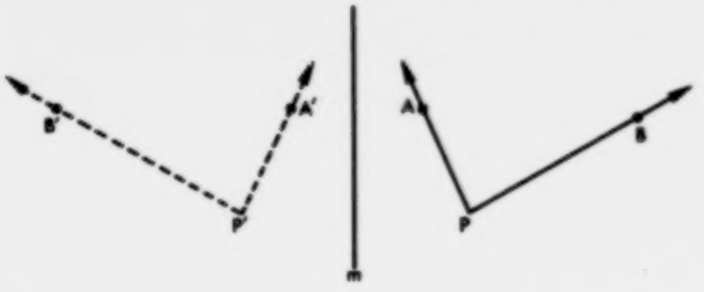


Do the rays at A, at B, or at C have the greatest spread? Which rays have the least spread?

One way of telling is to draw an arc of a circle across each ray, using in turn points A, B, and C as centers. Each arc should have the same radius (or opening of your compass). After the arcs are drawn, compare the distance between intersection points just as you did for the pie.

This method of comparing ray spreads may seem crude, but it can be very precise, especially, for smaller spreads. Later you will learn of another way to compare spreads by using a special instrument designed for this purpose.

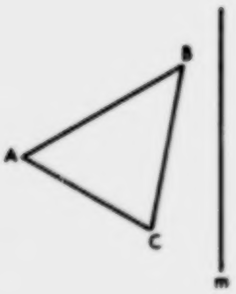
Activity 4: On a sheet of unlined paper, draw line m and a pair of rays $\overrightarrow{PA}$ and $\overrightarrow{PB}$ as shown:



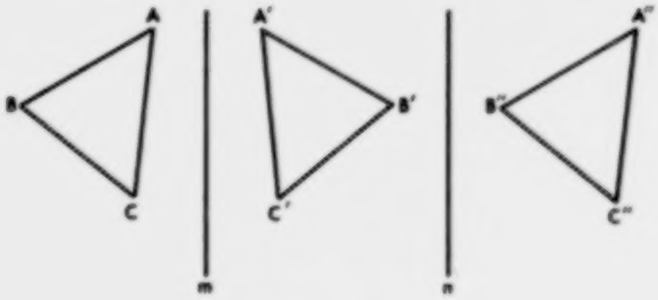
Find the reflections $\overrightarrow{P'A'}$ and $\overrightarrow{P'B'}$ of the rays $\overrightarrow{PA}$ and $\overrightarrow{PB}$ in m. Guess how the spreads of the rays at P and the rays at P′ compare. Check by using your compasses. What generalization seems to hold? Repeat the experiment with rays of a different spread.

Activity 5: On a sheet of unlined paper join 3 noncollinear points A, B, C.

The figure ABC is called "triangle ABC". Find the reflection of triangle ABC, in m. Compare the spreads of the rays at A, B, and C with those at A′, B′ and C′. How do the lengths of segments $\overline{AB}$, $\overline{BC}$, and $\overline{AC}$ compare with the lengths of their reflections, $\overline{A'B'}$, $\overline{B'C'}$, and $\overline{A'C'}$. Cut out Triangles ABC and A′B′C′. See if you can make them fit. Did you have to turn over one of the cutouts before making your figures fit? Will it always be necessary to turn over? If not, when will it be unnecessary?



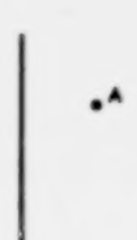
Activity 6: Now we are going to make a reflection and then a reflection of the image of this reflection, but in a different line. Draw the following on your unlined paper: Triangle ABC and lines m and n.



Find the reflection of ABC in m. Call it A′B′C′. Now find the reflection of A′B′C′ in n. Call this new figure A″B″C″. Your figures should look something like the figure above. Try to make some generalizations about the figures ABC, A′B′C′ and A″B″C″. Cut out the 3 figures. Do they fit? Should they fit well? Why do you think so?

9.8 Exercises

1. (a) Find the line containing point A that is perpendicular to line m. You may try folding your paper.



(b) Suppose now that A is on m, find the line containing point A that is perpendicular m. You may want to try folding your paper.

(c) Try to do (a) and (b) without folding.

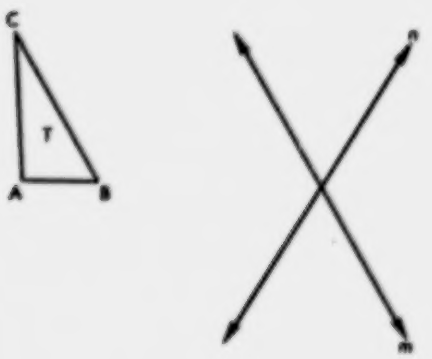
2. (a) What can you say about a triangle that has exactly one line of symmetry?

   (b) Can you find a triangle that has just two lines of symmetry?

   (c) Can you find a triangle that has just three lines of symmetry?

   (d) Are there triangles that have more than three lines of symmetry?

3.

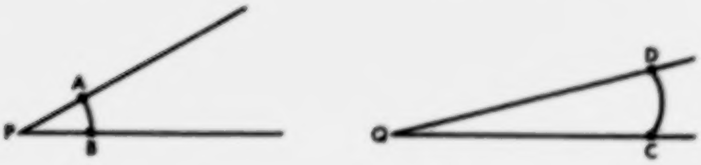


   (a) Find the reflection of Triangle T in m, call it "T_m" and the reflection of T_m in n, call it "T_{mn}", and finally, the reflection of T_{mn} in m, "T_{mnm}". Compare T, T_m, T_{mn}, T_{mnm}. What generalization would you care to make?

   (b) Carry out the same steps with m and n perpendicular lines. What can you say now that seems to be true?

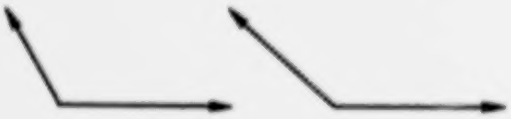
4. What is wrong in each of these cases?

   (a) The distance from A to B is less than the distance from C to D. Hence, the spread of the rays at P is less then the spread of the rays at Q.

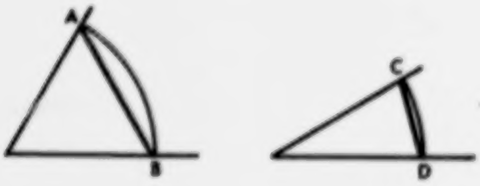


   (b) If two triangle cutouts fit then the spreads for three pairs of angles (one from each triangle) must be the same. Hence, if the spreads of pairs of angles for two triangles are the same, their cutouts should fit.

5. Why are comparisons difficult for the spreads of rays that are close to being opposite rays?



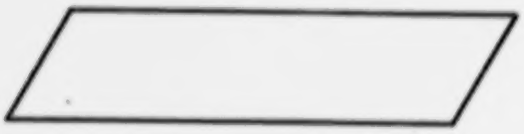
6. (a) If the distance from A to B is twice the distance from C to D, would you say that the spread for the first rays is twice the spread for the second?



   (b) Compare spreads for two opposite rays and a pair of perpendicular rays. Is your first spread twice as large as the second?

9.9 Symmetry In a Point

Does the parallelogram below have a line of symmetry?

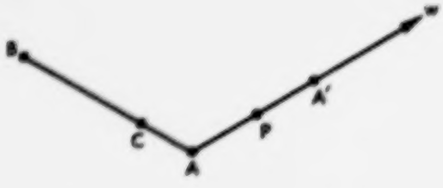


In other words, is there a line for which the parallelogram and its mirror image in this line are the same set of points?

After some experimentation, including folding, you will probably say that this parallelogram has no line of symmetry; there is no line reflection that leaves the parallelogram unchanged. However, as we shall soon see, the parallelogram does have a kind of symmetry; it is always symmetric in a point. Try to guess what symmetric in a point means.

Activity 7: Materials needed: Pencil, unlined paper, compass.

Let C be any point between points A and B. Let P be any point, not necessarily on line AB (See diagram below).



Draw a line through A and P, call it "w". Open your compass from P to A. With P as center and $\overline{PA}$ for radius, draw an arc crossing W in A', so that A,P,A' are in the same line, with P just as far from A as it is from A'. P is the midpoint of $\overline{AA'}$, and P bisects $\overline{AA'}$. We shall say that A' is the image of A under the symmetry in P. In the same way, find the image of B and C under the symmetry in P, calling the images B' and C' respectively.

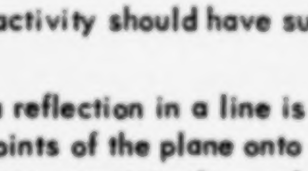
Are the points A', B', C' also collinear? Is C' between A' and B'? How does the distance from A to B compare with the distance from A' to B'? Compare the lengths AC with A'C', and BC with B'C'? What conjectures would you make from this activity regarding: collinearity of points, betweeness, isometry? Try to find a single line in which a reflection maps A into A' and B into B'.

The above activity should have suggested to you the following:

1. Just as a reflection in a line is a mapping of all the points of the plane onto all the points of the plane, symmetry in a point of a plane is also a mapping of all the points of the plane onto all the points of the plane.
2. Both mappings, reflection in a line and symmetry in a point,
   (a) are one-to-one,
   (b) are isometries,
   (c) map collinear points onto collinear points,
   (d) preserve betweeness.

What other properties would you conjecture? Perhaps the next activity will suggest some others.

Activity 8: Find the image of triangle ABC (usually written as "△ABC") under the symmetry in P. Call it △A'B'C' where A → A', B → B', C → C'.



Compare the spread of the rays of A, B, C with the corresponding spread of the rays at A', B', C'. How do the lengths AB, BC, and AC compare with A'B', B'C' and A'C'? What additional conjectures would you now make that have not been mentioned regarding the image of a line, ray, and segment under symmetry in a point? What conjecture would you make regarding the spread of two rays and the spread of their images under symmetry in a point?

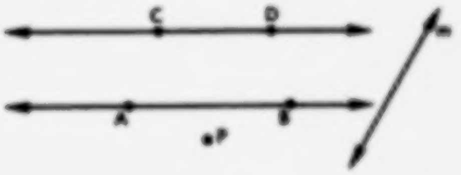
Have you thought of these:

3. Symmetry in a point, just as reflection in a line:
   (a) maps segments onto segments
   (b) rays onto rays
   (c) lines onto lines
   (d) preserves the spread of two rays

Cut out △ABC and △A'B'C'. Try to notice exactly what you have to do to make one triangle fit on the other. Do you have to turn one over before they will fit? Recall that for reflection in a line it was often necessary to turn over the figure or its image to obtain a fit.

The lines of your lined paper are parallel lines. In general, if two lines are in the same plane (flat surface) and do not cross, the lines are parallel. What happens to parallel lines under a reflection in a line and symmetry in a point?

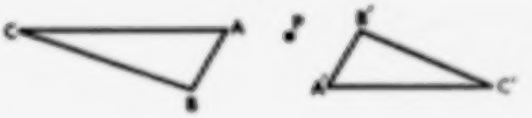
Activity 9: Materials needed: Pencil, lined paper, unlined paper, compass.



Draw a figure like the one shown here, with two lines parallel. Find the image of AB under a symmetry in P; call it A'B'. Does it seem that AB and A'B' are parallel? If they are parallel (let us abbreviate our writing by using the symbol "||" for "is parallel to") we have AB || A'B'. Find the image of CD under a symmetry in P, calling the image C'D'. Is CD || C'D'? What conjectures would you be willing to make now?

Find the reflections of the parallel lines CD and AB in m. Are the reflections parallel? Is CD parallel to its reflection in m? Have you made any of these conjectures?

1. A line maps onto a parallel line under symmetry in a point.
2. Two parallel lines map onto two parallel lines under symmetry in a point and reflection in a line.
3. The image of a figure under a symmetry in a point is a rotation of the figure through a "half turn":



9.10 Exercises

1. What point is its own image under a symmetry in point P?
2. Is there a point P in which a symmetry will map each of the following figures onto themselves? (If there is, show its location)
   (a) A line segment
   (b) A ray
   (c) A line

(d) A pair of parallel lines

(e) A parallelogram

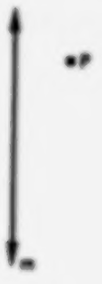
(f) The letter Z

3. If there is a point in which a symmetry will map a figure onto itself we say the figure is symmetric in a point. If there is a line in which a reflection will map a figure onto itself we say the figure is symmetric in a line. For each printed capital letter in the English alphabet, decide whether it is frequently symmetric in a point or in a line or neither.

Letter	Symmetric in a Point	Symmetric in a Line	Neither
A	No	Yes	–
B			
C			
.			
.			
.			
.			
Z			

4. Is there a line, m, in which a reflection will map each of the following figures onto itself? If there is, show it
   (a) A line segment
   (b) A ray
   (c) A line
   (d) A parallelogram

5. Using unlined paper and your compass obtain a line parallel to m. Hint: Find the image of m under the symmetry in P.



6. Try to find a way of obtaining a line through P parallel to m. (See Exercise 5)
   (a) by folding your paper
   (b) without folding but using your compass

7. What kind of symmetry does each of the following have?
   (a) A picture of a face (1) front view (2) side view
   (b) A circle
   (c) A square
   (d) A rectangle
   (e) A picture of a top
   (f) A picture of a 5 corner star
   (g) A picture of a 6 corner star
   (h) A swastika
   (i) A crescent

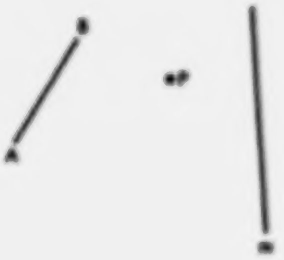
8. Denote by "pp" the symmetry mapping in point P, and by "ℓ_m" the reflection mapping in line m. Find the image of $\overline{AB}$ under each of the following composition mappings:

(a) ℓ_m with pp

(b) pp with ℓ_m

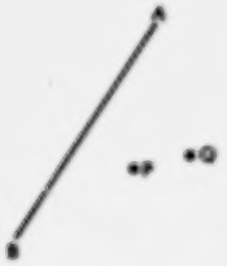
(c) ℓ_m with ℓ_m

(d) pp with pp



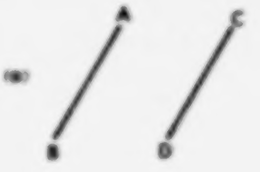
(e) ℓ_P with Q

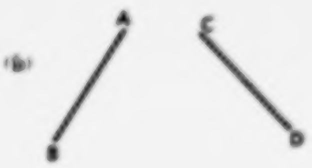
(f) ℓ_Q with ℓ_P

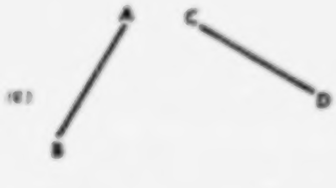


(g) Which of the above mappings (a-f) gave an image that was parallel to $\overline{AB}$? (We say that line segments are parallel if they are in parallel lines.)

9. If $\overline{AB}$ and $\overline{CD}$ have the same length, find one or more symmetries that will map $\overline{AB}$ onto $\overline{CD}$. (you may have to compose two symmetries)

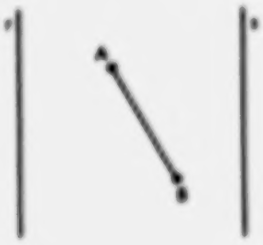






10. Let r || s. Find the image of $\overline{AB}$ under each of the following composition mappings:

(a) $\ell_r \circ \ell_s$ (b) $\ell_s \circ \ell_r$



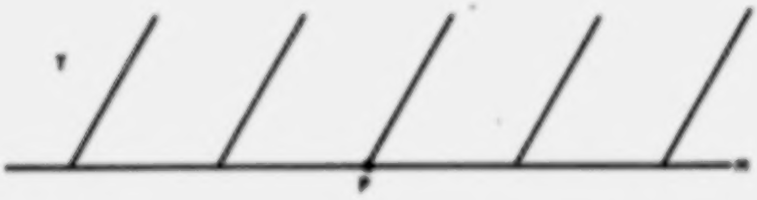
(c) Are the images found in (a) and (b)

(1) the same?

(2) parallel?

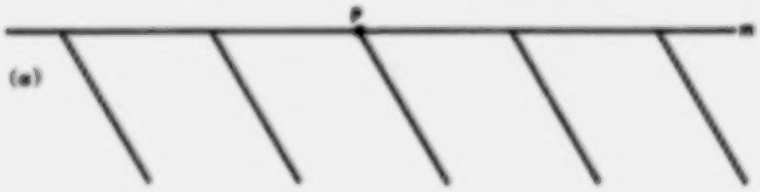
(3) parallel to $\overline{AB}$?

11. Consider the following design; call it T.

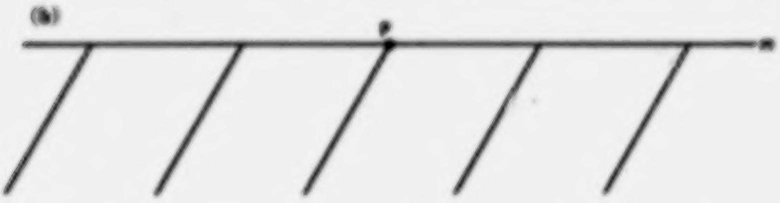


Describe how to obtain each of the following designs, using T or its images under mappings.

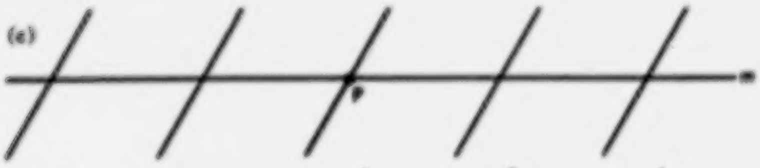
(a)



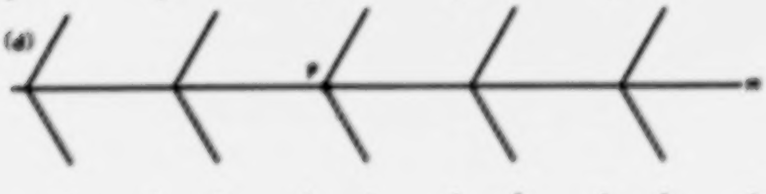
(b)



(c)



(d)



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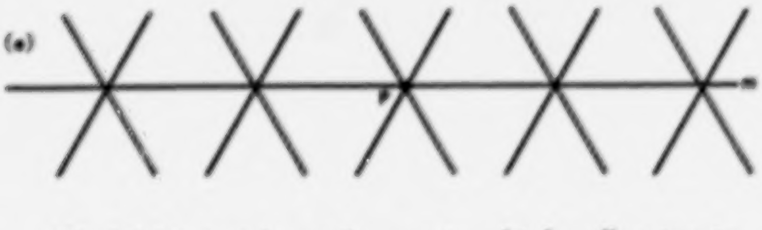
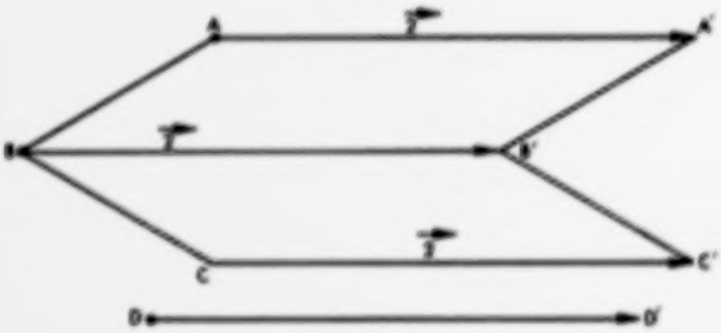


12. (a) List at least 5 ways in which reflection in a line and symmetry in a point are alike.

(b) List at least 2 ways in which they are not alike.

9.11 Translations

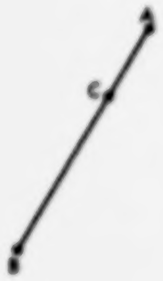
In chapter 4 we regarded $\overrightarrow{2}$ to be a mapping that sends every point of a plane onto a point of the plane 2 units to the "right" (or the "east"). Assuming that our unit is the inch, the mapping $\overrightarrow{2}$ of a few isolated points may be shown as follows:



$\overline{AB}$ is mapped onto $\overline{A'B'}$, $\overline{BC}$ is mapped onto $\overline{B'C'}$, D is mapped onto D'.

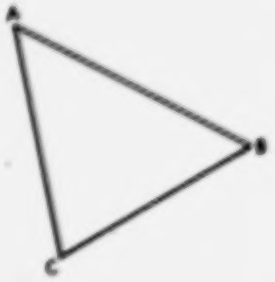
Activity:

Select points A, B, C on the parallel lines of your lined paper with C between A and B. Find the image of A, B, C under the translation $\overrightarrow{2}$.



Let the image of A, B, C be A', B', C'. Compare the distances AB with A' B', AC with A' C', BC with B' C'. How does the direction of $\overline{AB}$ compare with that of $\overline{A'B'}$? What can you say about $\overline{AA'}$, $\overline{BB'}$, $\overline{CC'}$? If C were the midpoint of $\overline{AB}$ what would you conjecture about C'?

Let A, B, C be non-collinear points on different lines of your paper. Find the image of △ABC under the translation $\overrightarrow{2}$. Call it △A' B' C'.



Compare the spreads of the rays at A with those at A', the rays at B with those at B', the rays at C with those at C'. What generalizations would you be willing to make for translations regarding: isometry, collinearity, betweeness, midpoints, parallelism, spreads? Carry out some other activity if you feel that you have to check some of your conjectures.

You may have thought of the following generalizations:

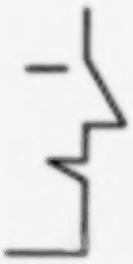
A translation

1. is an isometry
2. maps lines segments onto parallel line segments
3. preserves collinearity, betweeness and midpoints
4. preserves parallelism and spread

A translation need not have a magnitude of just two units and a direction only to the right. A translation may have a magnitude of any number of units and any fixed direction. Although our directed numbers showed 4 directions, there are in general infinitely many directions possible for a translation. Because we have the lines of our lined paper so handy, we shall be translating mainly to the right or left. However, one could always turn the paper so that a translation is along the parallel lines of our paper.

9.12 Exercises

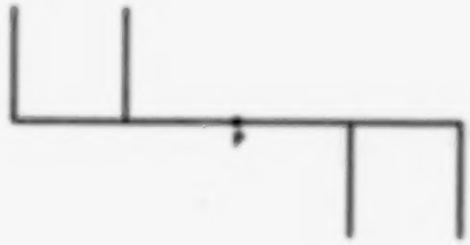
1. Which points, if any, are their own images under a translation?
2. Which of the following sets remain the same under some translation of magnitude greater than 0? Describe the translation(s).
   (a) segment
   (b) ray
   (c) line
   (d) plane
   (e) half-plane
3. Many designs are made by a succession of translations. You can make a face design by doing the following:
   (1) Draw a face on a blank sheet, about the size shown here, near the left edge of your paper.



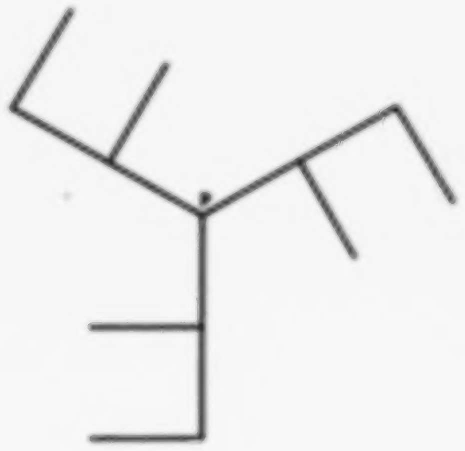
   (2) Place a piece of carbon paper face down on another blank sheet.
   (3) Mark off 2" intervals along the upper and lower edges of the paper under the carbon.
   (4) Line up the paper containing the face figure with the other paper.
   (5) Trace over the face figure with pencil.
   (6) Move face sheet 2" to the right using the marks you made as a guide and trace over face again.
   (7) Move face sheet 2" again to the right and trace face again.
   (8) You should be able to get 4 or 5 faces on your paper this way.
   (9) Try to describe the 4 or 5 faces in terms of translations.
4. Using the same face make 6 copies using the translation $1 + 1$ over and over.
5. Using a 2" square to start, make 6 copies using each of the following translations over and over:
   (a) $\overrightarrow{1}$ (b) $\overrightarrow{1} + 1\uparrow$
6. What happens when you use the same:
   (a) line reflection over and over on a figure and its image?
   (b) point symmetry over and over on a figure and its image?

9.13 Rotations

We have already observed that a point symmetry applied to a figure corresponds to giving the figure a half turn.



If we start with the figure to the left of P and apply the point symmetry pp we obtain the figure to the right of P. If we start with the figure to the right of P and apply pp we obtain the figure on the left of P. The entire figure above (the original F and its image under pp) is symmetric in P. But how would you regard the following figure?



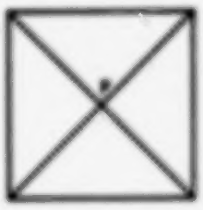
Is it symmetric in a line? in a point? It seems to have some kind of symmetry! If we rotate the figure $\frac{1}{3}$ of a complete rotation, we obtain the very same figure. Also, starting with any single F we can obtain the other two by rotating the figure through a $\frac{1}{3}$ turn twice. This suggests mappings which are rotations about some fixed point. A rotation in a point maps every point of the plane onto a point of the plane. What is needed to specify a rotation mapping?

We shall say that a figure has rotational symmetry if there is a point and a rotation, which is less than a full rotation but not a zero rotation, that maps the figure onto itself. Both F – figures above have rotational symmetry.

9.14 Exercises

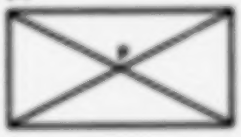
1. Which of the printed capital letters have rotational symmetry?
2. What properties are preserved under a general rotation like a $\frac{1}{3}$ turn? Which are not?
3. Let us denote by "$P_{\frac{1}{4}}$" a rotation that maps every point of the plane by a $\frac{1}{4}$ turn counterclockwise about point P. Which of the following figures are their own images under $P_{\frac{1}{4}}$?

(a)



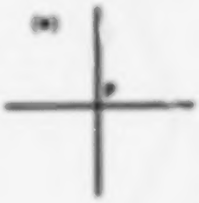
square

(c)

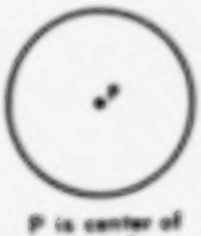


rectangle

(e)

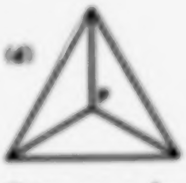


(b)



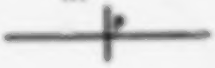
P is center of a circle

(d)



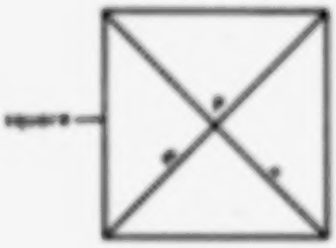
P is center of a triangle with sides of same length

(f)



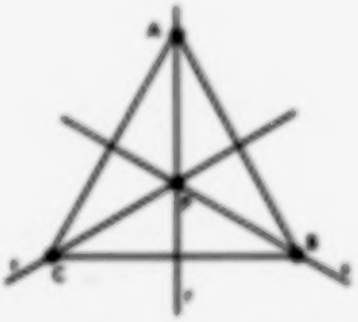
4. What kind of symmetry or symmetries does each of the following sets of points have?
   (a) Lattice Points of the First Quadrant
   (b) Lattice Points of the First and Second Quadrant
   (c) Lattice Points of the First and Third Quadrant
   (d) All the Lattice Points in a Plane
5. Let the operation be composition. Let e be the identity mappings: Fill in the following tables. ((a), (b), and (c) refer to the square. (d) refers to the triangle.)

	e	ℓ_m
e		
ℓ_m		



	e	ℓ_m	ℓ_n	p_P
e				
ℓ_m				
ℓ_n				
p_P				

	e	$P_{\frac{1}{4}}$	$P_{\frac{1}{2}}$	$P_{\frac{3}{4}}$
e				
$P_{\frac{1}{4}}$				
$P_{\frac{1}{2}}$				
$P_{\frac{3}{4}}$				



r, s, t are fixed lines on the plane. The lengths of $\overline{AB}$, $\overline{BC}$, and $\overline{AC}$ are the same.

	e	$P_{\frac{1}{3}}$	$P_{\frac{2}{3}}$	ℓ_r	ℓ_s	ℓ_t
e						
$P_{\frac{1}{3}}$						
$P_{\frac{2}{3}}$						
ℓ_r						
ℓ_s						
ℓ_t						

6. In 5(a)–(c), find the inverse for each of the mappings:
   (a) ℓ_m (b) p_P (c) $P_{\frac{1}{4}}$ (d) $P_{\frac{1}{3}}$ (e) $P_{\frac{1}{2}}$
7. Which mappings preserve
   (a) distances (d) midpoints
   (b) collinearity (e) direction of a line
   (c) betweeness (f) parallelism
   (g) clockwise orientation
8. Which mappings do not, in general, preserve
   (a) distances (d) midpoints
   (b) collinearity (e) direction of a line
   (c) betweeness (f) parallelism
   (g) clockwise orientation
9. Let us try to extend some of our mappings into 3 dimensions. Describe and try to give examples

of the corresponding symmetry for each of the following:

(a) Reflection in a plane

(b) Symmetry in a line (in space)

(c) Rotation about a line

(d) Translation in space

10. What are needed to specify each of the following types of mappings:

(a) A reflection in a line

(b) A symmetry in a point

(c) A translation

(d) A rotation

9.15 Summary of Chapter 9

1. A reflection in a line is a one-to-one mapping of all the points of a plane onto all the points of the plane preserving:

distance	midpoint
collinearity	spread
betweeness	parallelism

A reflection preserves neither orientation nor direction. If the reflection of A in m is A', then $\overline{AA'}$ is bisected by m. If m is the line in which a reflection is taken, then each point of m is its own image.

2. A symmetry in a point is a one-to-one mapping of all the points of a plane onto all the points of the plane preserving:

distance	spread
collinearity	parallelism
betweeness	orientation
midpoint	

A symmetry in a point maps a line onto a parallel line; it is the same as a half-turn. If the image of A under a symmetry in P is A', then P is the midpoint of $\overline{AA'}$. If P is the point in which a point symmetry is taken, then P is the only point that is its own image.

3. A translation is a one-to-one mapping of all the points of a plane onto all the points of the plane preserving:

distance	spread
collinearity	parallelism
betweeness	orientation
midpoint	

No point is its own image under a translation that has a magnitude greater than 0.

4. A rotation about a point is a one-to-one mapping of all the points of a plane onto all the points of the plane preserving:

distance	spread
collinearity	parallelism
betweeness	orientation
midpoint	

The point about which a rotation is taken is the only point that is its own image, unless the rotation is a multiple of a complete rotation.

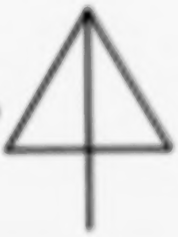
9.16 REVIEW EXERCISES

1. Fill in the table with "YES", if the mapping has the property, and "NO", if it does not.

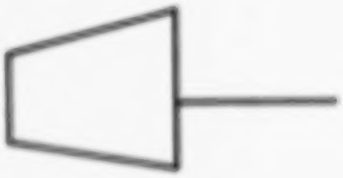
Mapping / Preserved	Reflection in a line	Symmetry in a line	Translation	Rotation
Distances (Isometry)				
Collinearity				
Betweeness				
Midpoint				
Spread				
Parallelism				
Orientation				

2. What kind of mapping and symmetry are suggested by each of the following

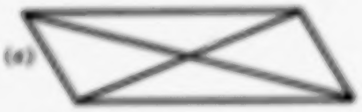
(a)



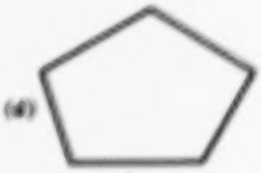
(b)



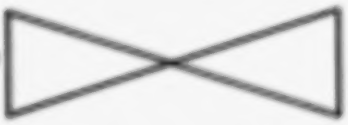
(c)



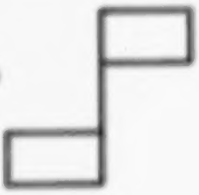
(d)



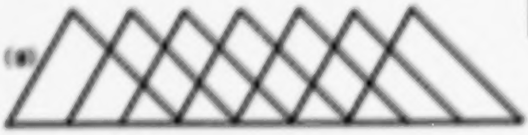
(e)



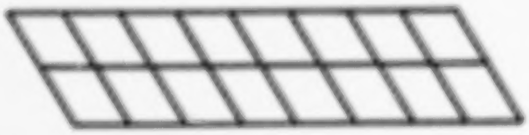
(f)



(g)



(h)



3. Which points are their own images under

(a) Reflection in a line

(b) symmetry in a point

(c) translation

(d) rotation

4. Which of the following figures may be identical with its image under one of the four mappings mentioned in Exercise 3?

Explain:

(a) line

(b) ray

(c) line segment

(d) two rays which are not opposite yet share a common end point.

(e) a square

(f) a rectangle

(g) a parallelogram

5. When are two lines perpendicular?

6. What holds for the two lines m and n if $\ell_m \circ \ell_n = \ell_n \circ \ell_m$?

7. Find all points P each of which has the same image under both composite mappings.

$$\ell_m \circ \ell_n \text{ and } \ell_n \circ \ell_m$$

8. What is the fewest line reflections whose compositions suffice to

(a) map any fixed point A onto a fixed point B?

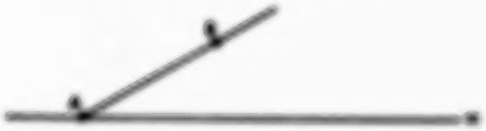
(b) map any fixed ray onto any fixed ray?

(c) map any fixed line onto any fixed line?

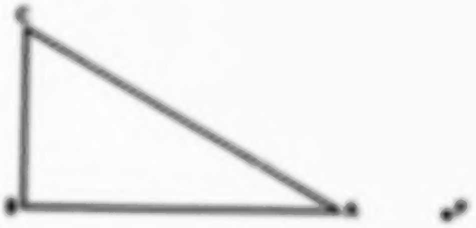
(d) map any fixed line segment onto any fixed line segment of the same length?

(e) map any $\triangle ABC$ onto $\triangle A'B'C'$ if $AB = A'B'$, $AC = A'C'$, $BC = B'C'$?

9. Find the reflection in m of $\overrightarrow{AB}$.



10. Find the image of $\triangle ABC$ under the symmetry in point P



11. In Exercise 10 apply $P_{\frac{1}{4}}$, $P_{\frac{1}{2}}$, $P_{\frac{3}{4}}$ to $\triangle ABC$.

CHAPTER 10
SEGMENTS, ANGLES, AND ISOMETRIES

10.1 Introduction

In previous chapters you have been introduced to many geometrical ideas which have been studied with the help of coordinates and mappings, particularly isometries. In this chapter, we shall tie together many of these results, make them more precise, and extend them to the study of angles.

Since isometries are distance preserving mappings, we shall look more closely at segments and their measure. Then we shall consider angles, how they are measured, and their behavior under an isometry. An interesting question will be whether or not the measure of an angle is preserved by an isometry.

We begin by considering some basic properties of lines and planes that are important for our study of segments and angles.

10.2 Lines, Rays, and Segments

It may seem to you, on reading this section, that we are making obvious statements and thus wasting time. If so, you will be confusing the obvious with the trivial. Obvious statements can have great significance. For instance, the statement: "The United States has only one president" is quite obvious, but its implications for the government and people of the United States are extremely important.

Our first statement about lines is obvious. It is called the Line Separation Principle and it expresses in a precise way the following idea: If we imagine one single point P removed from a line ℓ, the rest of the line "falls apart" into two distinct portions (subsets). Each of these portions is called an open halfline. Along each halfline, one can move smoothly from point to point without ever encountering point P. However, if one moves along line ℓ from a point in one halfline to a point in the other halfline, then it is necessary to cross through point P. See Figure 10.1.

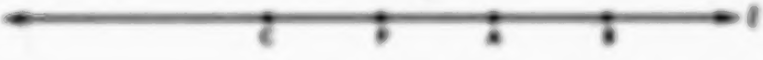


Figure 10.1

The mathematical way of stating this principle more precisely is as follows:

Any point P, on a line ℓ separates the rest of ℓ into two disjoint sets having the following properties:

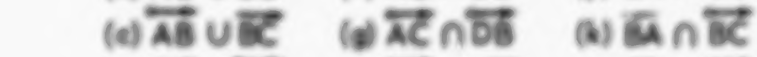
(1) If A and B are two distinct points in one of these sets then all points between A and B are in this set.

(2) If A is in one set and C is in the other, then P is between A and C.

One of these open halflines may be designated $\overset{\circ\rightarrow}{PA}$, the other $\overset{\circ\rightarrow}{PB}$. The little circle at the beginning of the arrow indicates that P itself is not a point of the open halfline. If P is added to $\overset{\circ\rightarrow}{PA}$ then we obtain the half-line, or ray, designated $\overrightarrow{PA}$ (no circle at the beginning of the arrow). You should be able to name two open halflines of ℓ with point A as the point of separation, and name two distinct rays starting at A. The starting point of a ray is called its vertex or end point. Note that $\overset{\circ\rightarrow}{PA}$ and $\overset{\circ\rightarrow}{PB}$ contain the same points, thus $\overset{\circ\rightarrow}{PA} = \overset{\circ\rightarrow}{PB}$; also $\overrightarrow{PA} = \overrightarrow{PB}$.

The set of points common to $\overrightarrow{PA}$ and $\overrightarrow{AP}$ is the segment $\overline{PA}$. Thus $\overrightarrow{PA} \cap \overrightarrow{AP} = \overline{AP}$. The set of points found in either PA or PC or both is the line ℓ. Thus $\overrightarrow{PA} \cup \overrightarrow{PC} = \ell$.

10.3 Exercise. In Exercise 1-3 refer to the line ℓ below.



1. Name two distinct rays of ℓ having C as end-point. Name the open halflines of ℓ for point of separation C.

2. Using two points name each of the following:

(a) $\overline{AB} \cup \overline{BC}$ (e) $\overline{AC} \cap \overline{DB}$ (i) $\overrightarrow{BA} \cap \overrightarrow{BC}$

(b) $\overrightarrow{AB} \cup \overrightarrow{BC}$ (f) $\overrightarrow{AC} \cap \overrightarrow{DB}$ (j) $\overrightarrow{BA} \cap \overline{BC}$

(c) $\overleftrightarrow{AB} \cup \overrightarrow{BC}$ (g) $\overrightarrow{AC} \cap \overleftrightarrow{DB}$ (k) $\overset{\circ\rightarrow}{BA} \cap \overrightarrow{BC}$

(d) $\overrightarrow{AB} \cup \overrightarrow{BC}$ (h) $\overrightarrow{AC} \cap \overline{BD}$ (l) $\overset{\circ\rightarrow}{BA} \cap \overrightarrow{BC}$

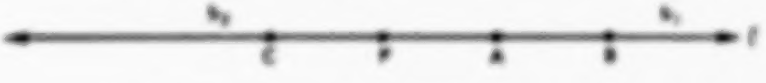
3. (a) Name a ray with endpoint B, containing E.

   (b) Name an open halfline contained in $\overrightarrow{BA}$. Are there others?

   (c) Describe the set of points determined by $\overrightarrow{CA} \cap \overrightarrow{AC}$.

   (d) Name a ray containing $\overline{BD}$. Are there others?

4. Let ℓ be a line and P one of its points. Let h_1 and h_2 be the two open halflines of ℓ determined by P. Let A and B be distinct points in h_1 and C a point in h_2. Determine whether each of the following statements is true or false;



(a) All points of $\overline{AB}$ are in h_1.

(b) All points of $\overrightarrow{AB}$ are in h_1.

(c) Either $\overrightarrow{AB}$ or $\overrightarrow{BA}$ contains C.

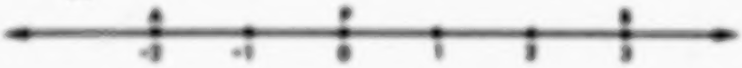
(d) Both $\overrightarrow{AB}$ and $\overrightarrow{BA}$ contain C.

(e) $\overrightarrow{PC}$ contains A

(f) $\overrightarrow{CP}$ contains A

(g) All points of $\overrightarrow{PB}$, other than P, are in h_1.

5.



Using the data shown in the above diagram tell what values x may have if x is the number assigned to a point in each of the following sets:

(a) $\overline{AB}$ (c) $\overrightarrow{BA}$ (e) $\overleftrightarrow{AB}$ (g) $\overrightarrow{AP} \cap \overrightarrow{PB}$

(b) $\overrightarrow{AB}$ (d) $\overrightarrow{AB}$ (f) $\overline{AB} \cap \overline{PB}$ (h) $\overrightarrow{AP} \cup \overrightarrow{PA}$

10.4 Planes and Halfplanes

A second separation principle concerns planes and is another example of an obvious statement. It states an essential property of planes.

It will help you to think about a plane if you imagine a very large flat sheet of paper, so large that its edges are inconceivably far and unreachable. In fact, it would be even better if you could think of a plane as having no edges, just as a line has no endpoints. In such a plane we could think of a line; otherwise a line, reaching any edge the paper might have, would have to stop and thus acquire an endpoint. But then it would not be a line!

We cannot draw a line, since any drawing would necessarily have to begin and end. In the same vein we cannot draw a plane. But we suggested a line by drawing a segment and arrows at each end. We suggest a plane by drawing a piece of it, as shown in Figure 10.2. Unfortunately there is no easy way to suggest in the drawing that the plane has no edges. However, to remind you that we are talking about a plane, rather, that a piece of it, we shall use script capital letters to name the plane. For instance, $\mathcal{P}$, $\mathcal{R}$, $\mathcal{S}$ will be names of planes.

Figure 10.2

Our second separation principle concerns planes. This Plane Separation Principle expresses in a precise manner, the following idea:

Any line ℓ in a plane $\mathcal{P}$ separates the rest of the plane into two distinct portions (subsets). Each of these portions is called an open halfplane. Within each halfplane one can move smoothly from point to point without ever encountering line ℓ. However, if one moves within plane $\mathcal{P}$ from a point in one open halfplane to a point in the other open halfplane, then it is necessary to cross line ℓ. The mathematical way of stating this is as follows:

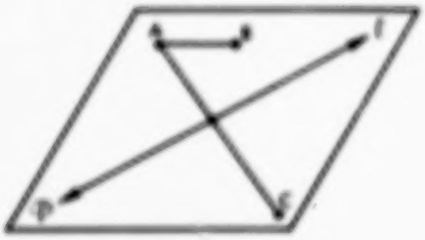


Figure 10.3

Any line ℓ in a plane $\mathcal{P}$ separates the rest of $\mathcal{P}$ into two disjoint sets having the following properties:

(1) If A and B are two distinct points in one of these sets then all points of $\overline{AB}$ are in this set.

(2) If A is in one set and C is in the other then $\overline{AC}$ (the segment, not $\overleftrightarrow{AC}$) intersects ℓ in a point.

The line ℓ is called the boundary of each open halfplane determined by ℓ, but actually it does not belong to either open halfplane. The union of an open halfplane with its boundary is called a halfplane.

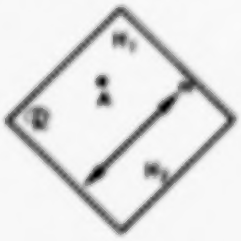
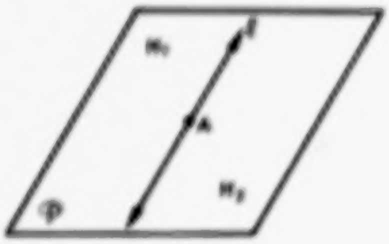


Figure 10.4

In the plane named $\mathcal{R}$ in Figure 10.4 you see line m separating $\mathcal{R}$ into the two halfplanes, named H_1 and H_2. If A is in H_1 w may also call H_1 the A-side of m. Then H_2 is the opposite side to the A-side.

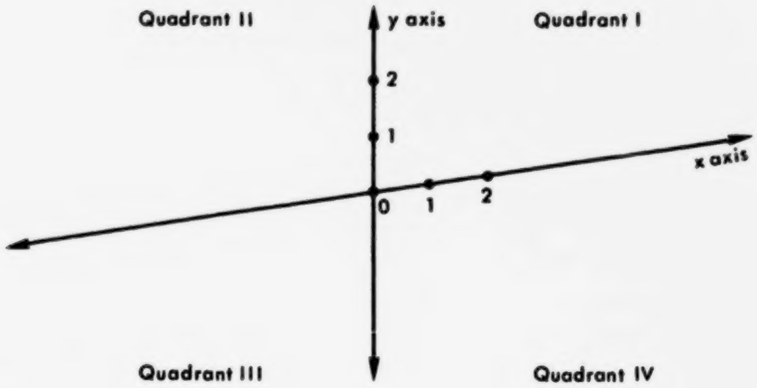
10.5 Exercises



Let $\mathcal{P}$ be a plane containing line ℓ and let ℓ contain point A. Let the two halfplanes determined by ℓ be H_1 and H_2. Determine whether each of the following statements is true or false:

1. Any line containing A, other than ℓ, contains points of H_1 and H_2.

2. Any ray with endpoint A, not lying in ℓ, contains points of H_1 and H_2.
3. Any segment containing A as an interior point, not lying in ℓ, contains points of H_1 and H_2.
4. If B and C are any two distinct points in H_1, not in ℓ, then $\overline{BC}$ intersects ℓ.
5. If B and C are any two distinct points in H_1, then $\overleftrightarrow{BC}$ does not intersect ℓ.
6. If B and C are two distinct points in H_2, then $\overleftrightarrow{BC}$ may not intersect ℓ.
7. If D is in H_1 and E is in H_2, then it is possible that $\overleftrightarrow{DE} \parallel \ell$.



8. The coordinate system shown separates the plane into four sets, each called a Quadrant. The x-axis separates the plane into two open halfplanes, one containing (0,2) the other containing (0, −2). Let us name the first of these open halfplanes H_{+x}, the other H_{-x}. Similarly, the y-axis separates the plane into two open halfplanes which we name H_{+y} and H_{-y}, with the obvious meaning attached to each. Now Quadrant I $= H_{+x} \cap H_{+y}$. In the same manner define Quadrants II, III, and IV.

10.6 Measurements of Segments

Let us examine what is involved when we use a ruler to find the length of a segment. We first place the graduated edge of the ruler against a line segment, say $\overline{AB}$, matching the zero point of the ruler with one of the points, say A.

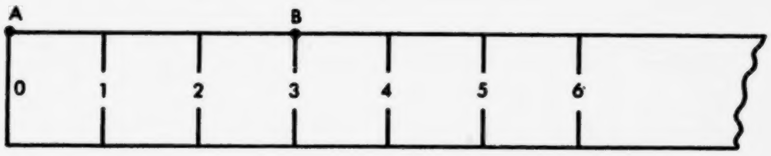


Figure 10.5

We then assign to point B the number on the ruler which matches it and say that the length of $\overline{AB}$, denoted by AB, is the number assigned to B. In our example the ruler assigns 0 to A and 3 to B. So AB = 3.

Now suppose we move the ruler to the left until it arrives at the position shown below.

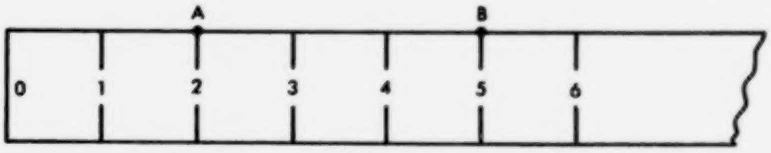


Figure 10.6

What is the number assigned by the ruler to A? to B? Using these numbers how can you find AB? Probably you subtracted 2 from 5 since this calculation gives the number of unit spaces in $\overline{AB}$. But suppose we turned the ruler around to this position.

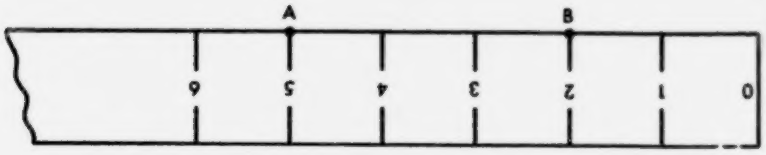


Figure 10.7

What are the assignments made by the ruler to A and B, in this position? Would you subtract 5 from 2 to find AB? This, of course, gives −3. In measuring the length of a segment we want to know how many unit spaces it contains. Therefore, we use only positive numbers for lengths of segments. If we do subtract 5 from 2, we must take the absolute value of the difference. In general, then, if a ruler assigns the numbers x_1 and x_2 to the end points of a segment $\overline{AB}$, we can use the distance formula

$$AB = |x_1 - x_2|$$

Let us now consider a ruler which has negative numbers on it (like a thermometer) that is placed against $\overline{AB}$ and looks like this,

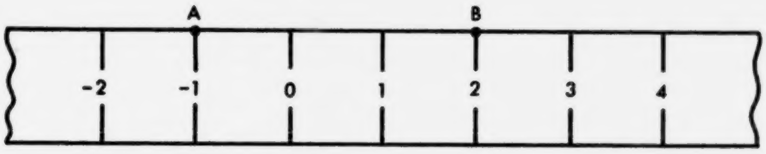


Figure 10.8

or perhaps like this,

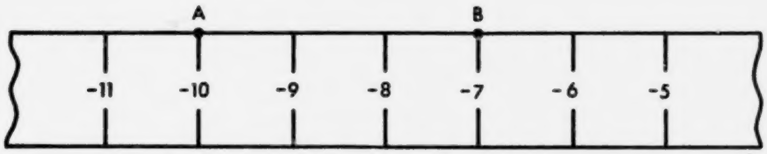


Figure 10.9

or even like this.

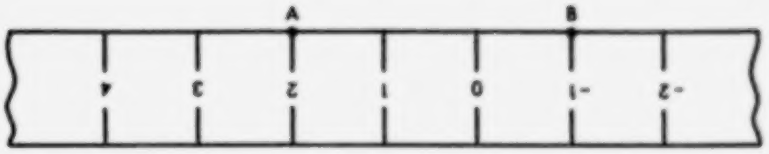


Figure 10.10

Does the distance formula give us the number of unit spaces in each case? Let us see.

For the fourth position (Figure 10.8) the formula yields: $AB = |-1-2|$

For the fifth position (Figure 10.9) the formula yields: $AB = |-10-(-7)|$

For the sixth position (Figure 10.10) the formula yields: $AB = |2-(-1)|$

Is 3 the value of AB in each case?

You know that the distance from A to B should be the same as the distance from B to A. In the formula this reverses x_1 and x_2. Is it true that $|x_1 - x_2| = |x_2 - x_1|$?

Let us review the results of this section in terms of mappings.

(a) A ruler assigns numbers x_1 and x_2 to the endpoints of $\overline{AB}$. Thus $A \rightarrow x_1$ and $B \rightarrow x_2$. Then we say $AB = |x_1 - x_2|$.

(b) Moving the ruler 2 spaces to the left (as we did) is a translation with rule $n \rightarrow n+2$. Thus $x_1 \rightarrow x_1 + 2$ and $x_2 \rightarrow x_2 + 2$. We ask you to answer two questions:

(1) Does a translation preserve distance?

(2) Is $|x_1 - x_2|$ preserved under this translation?

Suppose the ruler were moved to the right. Are the last two answers changed?

(c) In the fourth and fifth positions (Figures 10.8 and 10.9) we moved the ruler still further to the left. Is the composition of two translations still a translation? Do the answers to our two questions change for the third position?

(d) Let us compare the rulers in the first and last positions.

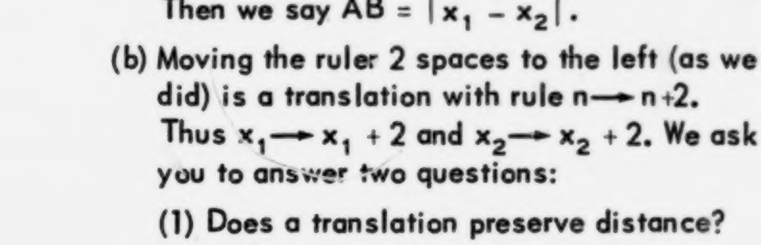
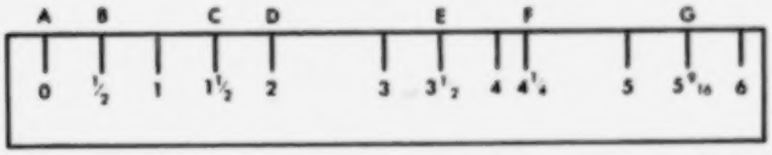


Figure 10.11

Do you see a mapping of Z into Z with the rule $n \rightarrow 4 - n$? Then $x_1 \rightarrow 4 - x_1$ and $x_2 \rightarrow 4 - x_2$. But $|(4-x_1) - (4-x_2)| = |x_2 - x_1|$. And again we can say yes to our two questions above. We conclude that the distance formula gives the correct distance for all positions of a ruler.

10.7 Exercises

1. In this exercise use the numbers assigned by the ruler to points in the diagram below. First express the length of the segments listed below in the form $|x_1 - x_2|$. Then compute the length.



(a) $\overline{AC}$
(b) $\overline{AE}$
(c) $\overline{AG}$
(d) $\overline{FA}$
(e) $\overline{BC}$
(f) $\overline{BD}$
(g) $\overline{FB}$
(h) $\overline{GB}$
(i) $\overline{CD}$
(j) $\overline{FC}$
(k) $\overline{EF}$
(l) $\overline{GF}$

2. A ruler, graduated with negative and positive numbers assigns 0 to point A. What number does it assign to B if AB = 3? (Two answers)

3. A ruler assigns 8 to D. What number does it assign to E if DE = 2. (Try to solve this problem by solving the equation $|x-8| = 2$.)

4. A ruler assigns 83 to F. What number can it assign to G if $FG = 6\frac{1}{2}$?

10.8 Midpoints and other Points of Division

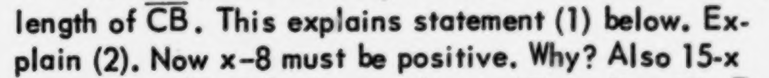


Figure 10.12

Let a ruler assign 8 to A and 15 to B. We shall try to find the number assigned to C, the midpoint of $\overline{AB}$. Let that number be represented by x (See Figure 10.12). You recall that a midpoint of a segment bisects it. This means that the length of $\overline{AC}$ is the same as the length of $\overline{CB}$. This explains statement (1) below. Explain (2). Now x-8 must be positive. Why? Also 15-x is positive. Why? So the equality in (2) implies (3). Explain (4) and (5). Check whether for $x = 11\frac{1}{2}$, AC = CB.

(1) $AC = CB$
(2) $|x-8| = |15-x|$
(3) $x-8 = 15 - x$
(4) $2x = 23$
(5) $x = 11\frac{1}{2}$

Use this method of finding the number assigned to a midpoint of $\overline{DE}$ if a ruler assigns -2 to D and 5 to E.

Let us generalize this method, that is, let us find a formula for midpoints. Let a ruler assign x_1 to A and x_2 to B where $x_1 < x_2$ and let x represent the number assigned to C, the midpoint of $\overline{AB}$ (See Figure 10.13).

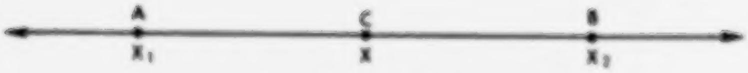


Figure 10.13

Then,

$$AC = CB$$
$$|x - x_1| = |x_2 - x|$$
$$x - x_1 = x_2 - x \quad \text{(Why?)}$$
$$2x = x_1 + x_2$$
$$x = \tfrac{1}{2}(x_1 + x_2)$$

Do you recognize that x is the mean of x_1 and x_2? This is an easy way to remember this formula.

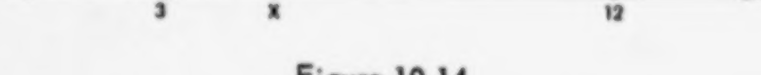


Figure 10.14

Suppose R is in $\overline{PQ}$ and it divides $\overline{PQ}$ in the ratio 1:2 from P to Q. (The phrase "from P to Q" tells that PR corresponds to 1 and RQ to 2.) To find x for the data shown in Figure 10.14 we can proceed as follows:

(1) $\dfrac{|x-3|}{|12-x|} = \dfrac{1}{2}$ or $2|x-3| = |12-x|$

Both x−3 and 12−x are positive.

(2) $2 \cdot (x-3) = 12-x$

(3) $2x-6 = 12-x$ Check $\dfrac{|6-3|}{|12-6|} = \dfrac{1}{2}$

(4) $3x = 18$

(5) $x = 6$

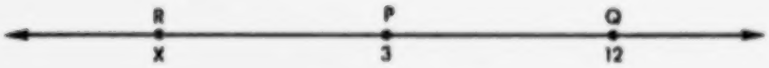


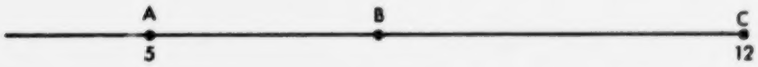
Figure 10.15

Suppose, instead, that R were not between P and Q. Then 3−x is positive and 12−x is positive. Then step (2) above becomes (2′) $2 \cdot (3-x) = (12-x)$. Complete the solution and check.

10.9 Exercises

In exercises 1-4 you are asked to derive results which are going to be used in later developments. In this respect they differ from other exercises whose results can be forgotten without harm to an understanding of future developments. These exercises are starred (*). In following sections such exercises will also be starred.

*1. Let B be an interior point of $\overline{AC}$ and let a ruler assign numbers 5 and 12 to A and C, as shown.



(a) What is one possible assignment to B that guarantees that B is an interior point of AC. Name three other possible assignmentts to B that also guarantee that B is between A and C. What are all the possible assignmeents to B such that B is between A and C?

(b) Show that AB + BC = AC if B is assigned the number 8 or the number $11\frac{1}{2}$.

(c) Show that AB + BC = AC if B is assigned the number x such that $5 < x < 12$.

This last result may be stated in general as follows: If B is between A and C, then AB + BC = AC. It is called the

*2. Additive Property of Betweeness for Points.

Suppose two circles in a plane have centers at A and B, and respective radii r_1 and r_2. We are going to compare AB with $r_1 + r_2$ for different positions of the two circles.

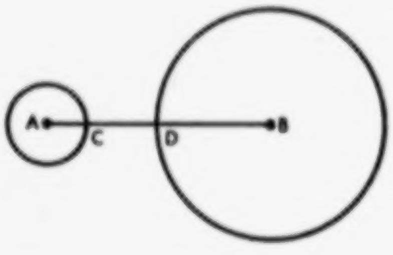


Figure 10.16

(a) Suppose the circles do not intersect as shown in Figure 10.16. Then AB = AD + DB (Why?) and AD = AC + CD (Why?) So AB = AC + CD + DB. But AC = r_1 and DB = r_2. Hence AB = r_1 + CD + r_2. Thus AB > $r_1 + r_2$.

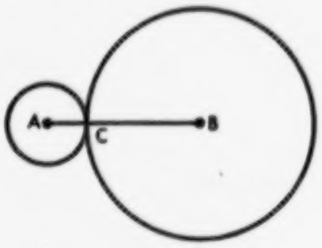


Figure 10.17

(b) Consider the position of the circles in Figure 10.17, in which the circles just touch at C. Show that AB = $r_1 + r_2$.

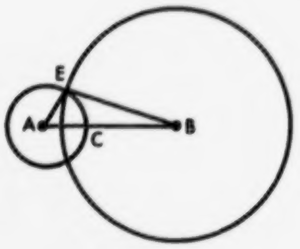


Figure 10.18

(c) Consider the position of the circles in Figure 10.18 in which they intersect. One of the points of intersection is named E.

Now $AB = AC + CB$, Why? and $CB < r_2$

so $AB < r_1 + r_2$, Why?

$\overline{EA}$ and $\overline{EB}$ are also radii and therefore $EA = r_1$ and $EB = r_2$.

Therefore $AB < EA + EB$

In words, this last result means, that the length of any side of a triangle ($\triangle ABE$ in this case), is less than the sum of the lengths of the other two. We call this conclusion the Triangle Inequality Property. You should note that for any triangle, there are three inequalities. Thus, for $\triangle DEF$ (Figure 10.19)

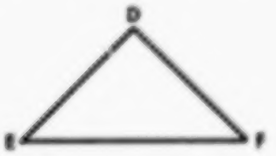


Figure 10.19

(a) $DE < EF + FD$

(b) $EF < FD + DE$

(c) $FD < DE + EF$

3. For Figure 10.20, we see by the triangle Inequality Property that in $\triangle ABD$, $DA + AB > DB$. Use this fact to show that the perimeter of $\triangle DAC$ is greater than the perimeter of $\triangle DBC$.

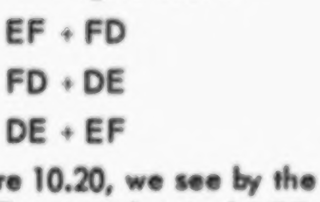


Figure 10.20

4. Show in $\triangle ABC$ that the difference between the lengths of any two sides is less than the length of the third side.

5. Which of the following triplets of numbers may be the lengths of the sides of a triangle?

(a) 5, 6, 8 (d) 4.1, 8.2, 12.3

(b) 5, 6, 11 (e) 18, 22, 39

(c) 1, 2, 3 (f) $4\frac{1}{2}$, $4\frac{3}{4}$, $4\frac{5}{8}$

10.10 Using Coordinates to Extend Isometries.

Let us consider an isometry, f, of a pair of points $\{A, B\}$. If $f: A \rightarrow A'$ and $B \rightarrow B'$, then $AB = A'B'$. How can we extend this isometry to a third point of $\overleftrightarrow{AB}$? This is easily done by working with the line coordinate system on $\overleftrightarrow{AB}$ that assigns 0 to A and 1 to B. Since $AB = A'B' = 1$, there is a coordinate system on $\overleftrightarrow{A'B'}$ that assigns 0 to A′ and 1 to B′.

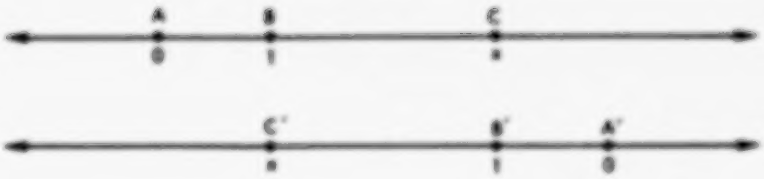


Figure 10.21

Now suppose C is any point on $\overleftrightarrow{AB}$ and let its coordinate be x. We can extend f to C by taking for its image the point C′ on $\overleftrightarrow{A'B'}$ whose coordinate is also x. To convince yourself that we have succeeded in extending f you should verify that $AC = A'C'$ and $BC = B'C'$. You can do this by using the distance formula. How can you extend f to other points of $\overleftrightarrow{AB}$?

Let us go on to consider an isometry, g, of three noncollinear points $\{A, B, C\}$ and how to extend g to a fourth point in the plane of A, B, C.

Draw a triangle with plane coordinates as shown in Figure 10.22. On another paper trace $\triangle ABC$, calling it $\triangle A'B'C'$, and giving A′, B′, C′, the same coordinates respectively as A, B, C. Take any point D on the first paper and read its coordinates. Locate the point D′ on the second paper with the same coordinates as D. Now place one paper over the other so that $A \rightarrow A'$, $B \rightarrow B'$, $C \rightarrow C'$. Does $D \rightarrow D'$? What conclusion seems indicated from this experiment? How can you extend g to other points of the plane?

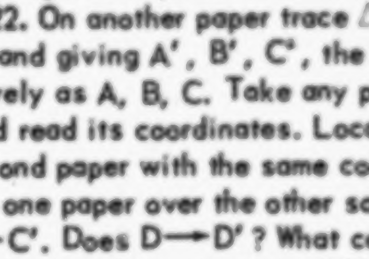


Figure 10.22

10.11 Coordinates and Translations

As you will see, coordinates are quite useful in studying translations of points of a plane onto points of the same plane. Suppose point A has coordinates (1, 3) in some plane coordinate system and is mapped onto A′, with coordinates (4, 5) by a translation. We can regard this translation as the composition of two motions. The first, moves a point 3 units in the direction of the positive x-axis and is followed by a second motion of 2 units in the direction of the positive y-axis. Any other point of the plane will also have an image under this composite translation. The rule of this translation is easy to write.

$$x \longrightarrow x+3$$

$$y \longrightarrow y+2$$

or simply $(x,y) \longrightarrow (x+3, y+2)$

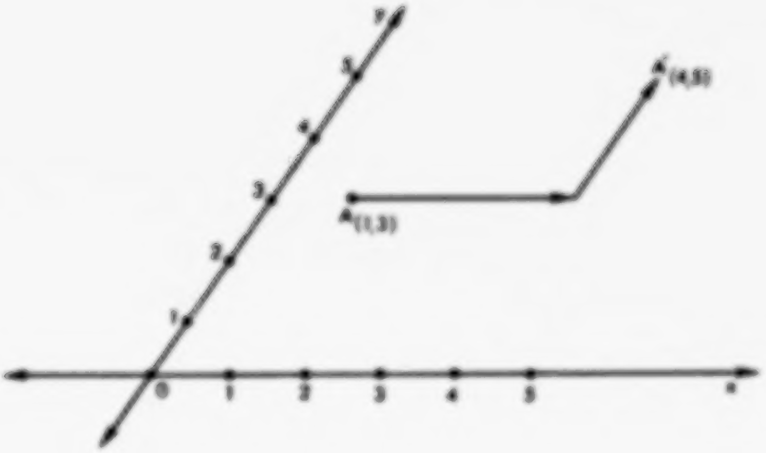


Figure 10.23

Under this rule B with coordinates (3, 8) is mapped onto B′ with coordinates (6, 10).

Now consider ABB′A′ in Figure 10.24. Under the translation above $\overline{AB} \longrightarrow \overline{A'B'}$. This leads to the conclusion that AB = A′B′ and $\overleftrightarrow{AB} \parallel \overleftrightarrow{A'B'}$. Thus ABB′A′ is a parallelogram.

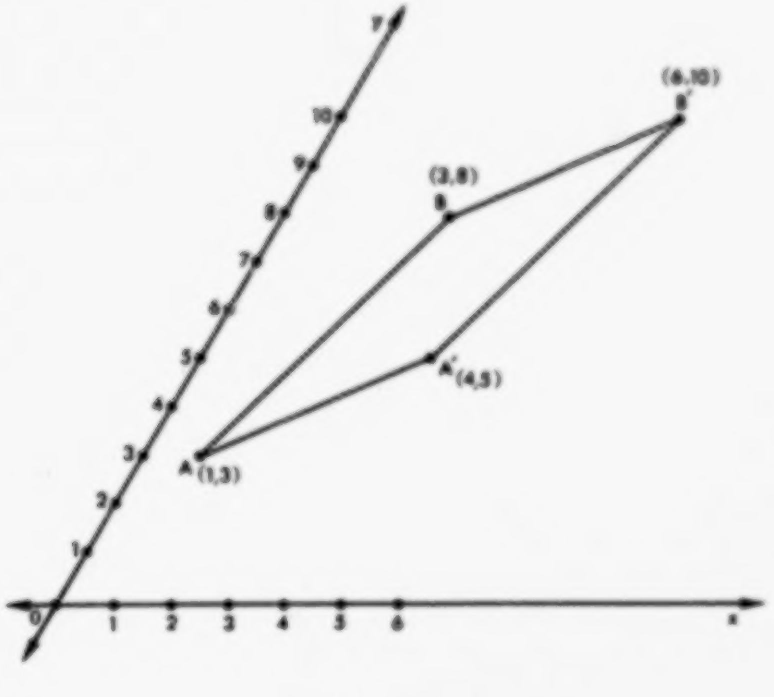


Figure 10.24

We can now check some old results about parallelograms in terms of coordinates, in particular, whether the diagonals bisect each other. But the coordinate formula for midpoints available to us is for line coordinates. We must therefore develop a formula for plane coordinates.

In Figure 10.25 we show only the diagonal $\overline{AB'}$. Let M be the midpoint of $\overline{AB'}$ and consider the parallel projection in the direction of the y-axis onto the x-axis. This projection maps A onto A′, M onto M′ and B onto B′. Since a parallel projection preserves midpoints it follows that M′ is the midpoint of $\overline{A'B'}$. But the line coordinate of M′ is $\frac{1}{2}$ (1+6) or $\frac{7}{2}$. Since M obtains its x-coordinate by parallel projection then the x-coordinate of M is $\frac{7}{2}$. Using a diagram, show that the y-coordinate of M is $\frac{1}{2}$ (3+10) or $\frac{13}{2}$.

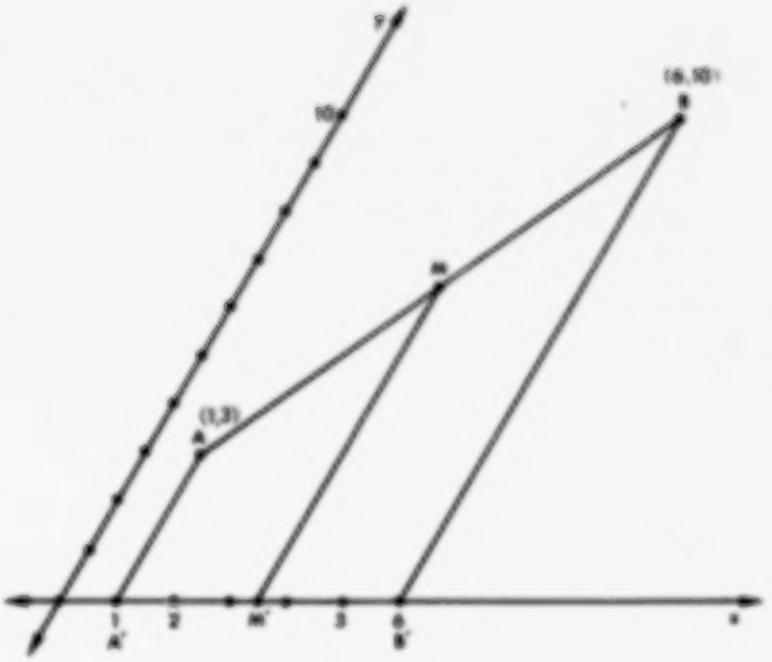


Figure 10.25

In general, if P has coordinates (x_1, y_1) and Q has coordinates (x_2, y_2) then the midpoint of PQ has coordinates.

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

Now verify that the coordinates of the midpoint of $\overline{BA'}$ are also $(\frac{7}{2}, \frac{13}{2})$. Does this verify that the diagonals of ABB′A′ bisect each other?

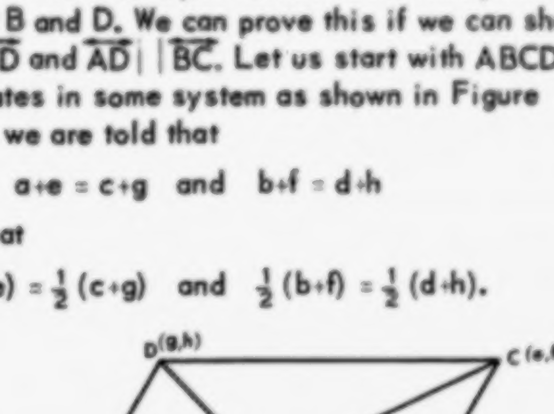
There is a bonus in this consideration, which you will be asked to prove in an exercise. It is this: In any parallelogram the sum of the x-coordinates of either pair of opposite vertices is the same. In fact we can go on to say that ABCD is a parallelogram if the sum of the x-coordinates of A and C is equal to the sum of the x-coordinates of B and D, and the sum of the y-coordinates of A and C is equal to the sum of the y-coordinates of B and D. We can prove this if we can show that $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$ and $\overleftrightarrow{AD} \parallel \overleftrightarrow{BC}$. Let us start with ABCD and coordinates in some system as shown in Figure 10.26. Then we are told that

$$a+e = c+g \quad \text{and} \quad b+f = d+h$$

It follows that

$$\tfrac{1}{2}(a+e) = \tfrac{1}{2}(c+g) \quad \text{and} \quad \tfrac{1}{2}(b+f) = \tfrac{1}{2}(d+h).$$

Figure 10.26

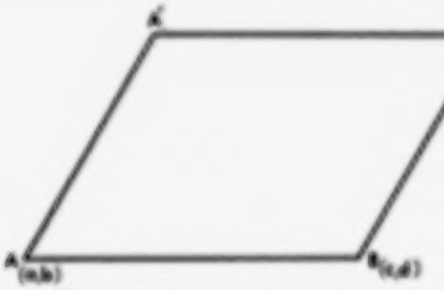


This means that $\overline{AC}$ and $\overline{BD}$ bisect each other, say in M. Thus M is the center of a point symmetry that maps A onto C and B onto D.

Point symmetry preserves parallelism. Hence $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$. M is also the center of a point symmetry that maps A onto C and D onto B thus $\overleftrightarrow{AD} \parallel \overleftrightarrow{BC}$. We conclude that ABCD is a parallelogram.

10.12 Exercises

1. Let ABB'A' be a parallelogram. It can be regarded as having been formed by a translation



under which A⟶A' and B⟶B'. Suppose A and B have coordinates (a, b) and (c, d) respectively in some coordinate system. Let the translation have the rule:

$x \longrightarrow x + p$ and $y \longrightarrow y + q$.

Then A' has coordinates (a+p, b+q) and B' has coordinates (c+p, d+q).

(a) Using the midpoint formula show that $\overline{AB'}$ and $\overline{A'B}$ bisect each other.

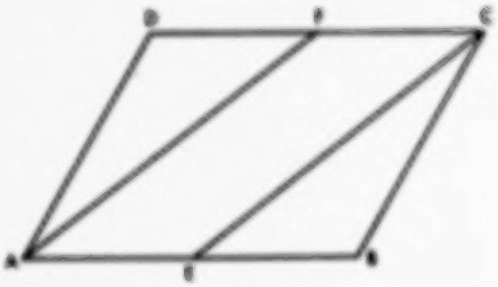
(b) Show that the sum of the x-coordinates of A and B' is equal to the sum of the x-coordinates of A' and B.

(c) Show that the sum of the y-coordinates of A and B' is equal to the sum of the y-coordinates of A' and B.

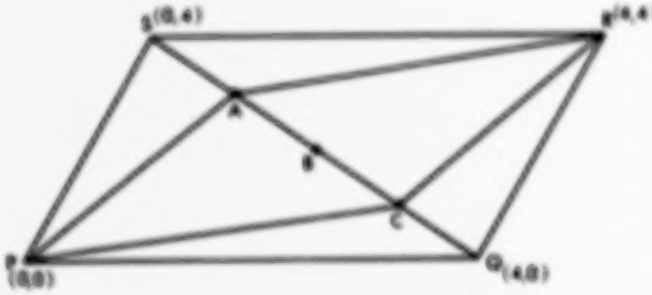
2. Suppose ABCD is a parallelogram and the coordinates of three vertices are given. Find the coordinates of the missing vertex. Check your answers with a drawing.

(a) A (0,0)	B (3,0)	D (0,2)
(b) A (0,0)	B (3,2)	D (2,3)
(c) A (2,1)	B (5,6)	C (0,0)
(d) A (3,2)	C (-3,2)	D (-2,5)
(e) B (-3,2)	C (3,3)	D (2,5)
(f) A (0,0)	B (a,0)	D (0,b)
(g) A (a,b)	B (c,d)	C (e,f)

3. Suppose ABCD is a parallelogram, that E is the midpoint of $\overline{AB}$ and F is the midpoint of $\overline{CD}$. Show that AECF is also a parallelogram. (You can simplify the proof by using the coordinate system in which A, B, D have coordinates (0,0), (1,0) and (0.1) respectively).



(a) Using the indicated coordinates, show that PQRS is a parallelogram.



(b) Suppose B is the midpoint of $\overline{SQ}$, that A is the midpoint of $\overline{SB}$ and C is the midpoint of $\overline{BQ}$. Show that PCRA is also a parallelogram.

5. For the parallelogram PQRS in Exercise 4 take any suitable coordinates for the vertices and show again that PCRA is a parallelogram. What is the significance of taking any suitable coordinates for P, Q, R, S?

6. Using coordinates, show that translations preserve midpoints.

10.13 Perpendicular Lines

In a preceding chapter we studied reflections in a line. In this section we use such reflections to review and extend the idea of perpendicular lines.

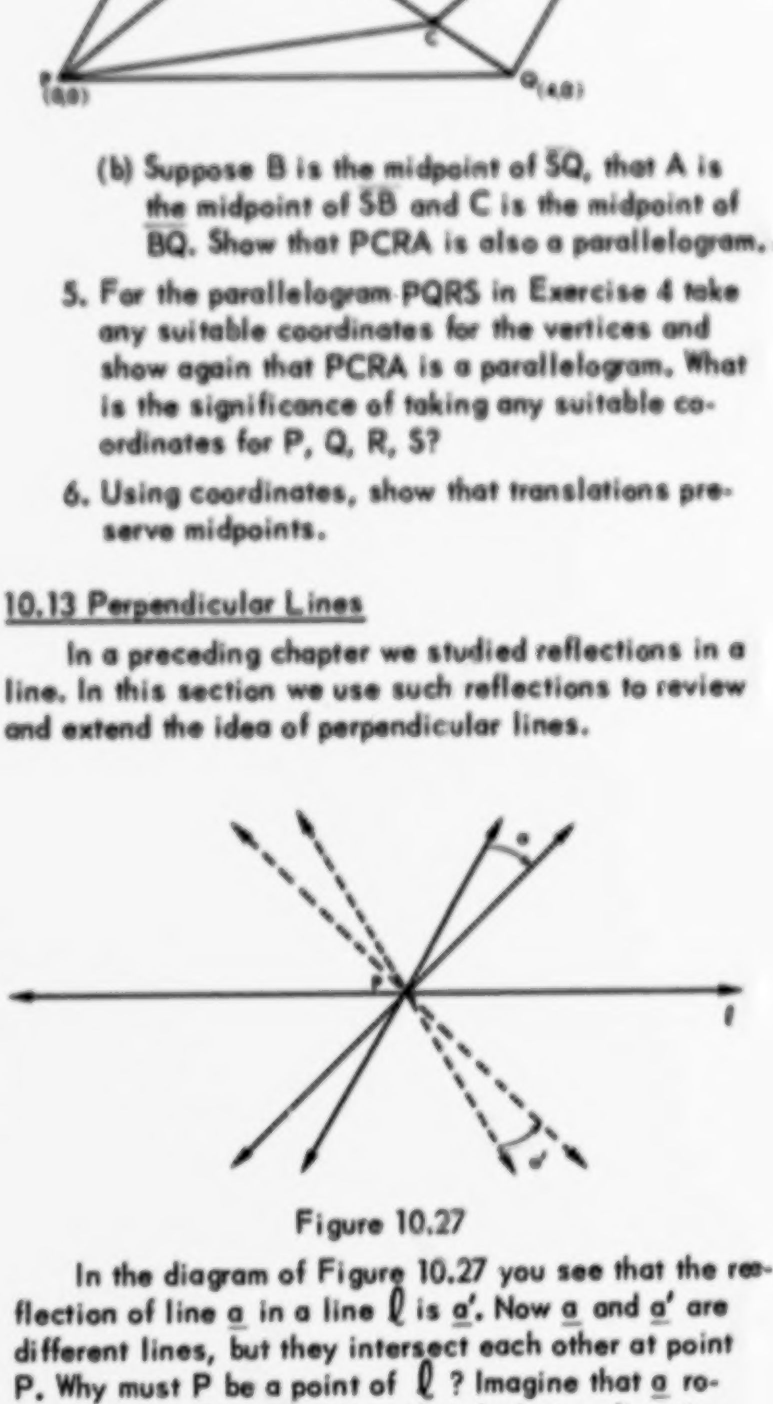


Figure 10.27

In the diagram of Figure 10.27 you see that the reflection of line a in a line ℓ is a'. Now a and a' are different lines, but they intersect each other at point P. Why must P be a point of ℓ? Imagine that a rotates around P as a pivot in the clockwise direction. Let a' continue to be the reflection of a. How does a' rotate? In the course of these rotations, does a' ever become the same as a?

Now rotate $\underline{a}$ in a counterclockwise direction. In the course of this rotation does $\underline{a}'$ again become the same as $\underline{a}$?

We see that $\underline{a}$ can be its own image, as it rotates about P, in two ways. In one of these $\underline{a} = \ell$; in the other $a \neq \ell$. For the second case $\underline{a}$ is perpendicular to ℓ. In general two lines are perpendicular if they are different lines, and one of them is its own image under a line reflection in the other.

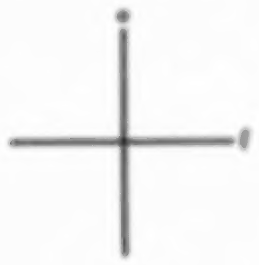


Figure 10.28

We denote that $\underline{a}$ is perpendicular to ℓ by writing $\underline{a} \perp \ell$. Note that ℓ is also its own image under a reflection in $\underline{a}$ (Figure 10.28). So $\ell \perp \underline{a}$ whenever $\underline{a} \perp \ell$. Also note that the plane is separated by each of the two perpendicular lines into two halfplanes and that any point in one of these halfplanes has its own image in the other.

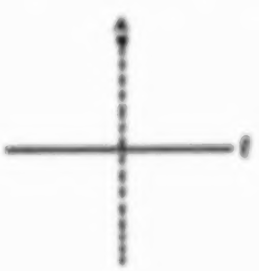
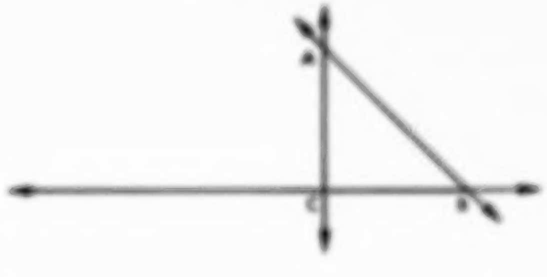


Figure 10.29

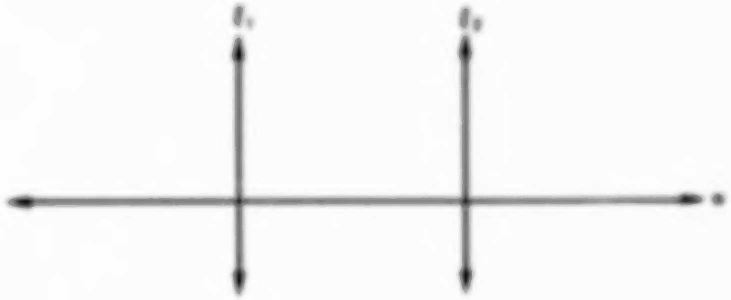
On a piece of paper draw line ℓ and mark a point A, either on or off ℓ, as in Figure 10.29. Fold the paper along a line containing A such that one part of ℓ falls along the other. In how many ways can this fold be made? You know that the line of the crease is perpendicular to ℓ. It would seem then that there is exactly one line containing a given point that is perpendicular to a given line.

10.14 Exercises

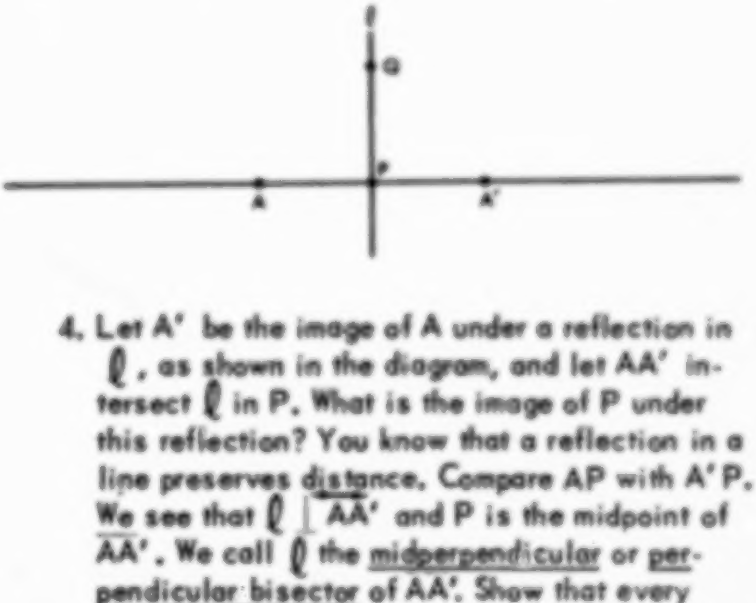
1. For this exercise draw two parallel lines on your paper, calling them $\underline{a}$ and $\underline{b}$.
   (a) Fold the paper so that one part of $\underline{a}$ falls along the other part. Label the crease $\underline{c}$. Is $\underline{c} \perp \underline{a}$? Why?
   (b) For the fold you made in (a), does part of $\underline{b}$ fall along another part of itself? What bearing does your answer have on the perpendicularity relation of $\underline{c}$ and $\underline{b}$?
   (c) Tell how the results of this experiment support or do not support this statement: If two lines are parallel, a line perpendicular to one is perpendicular to the other.



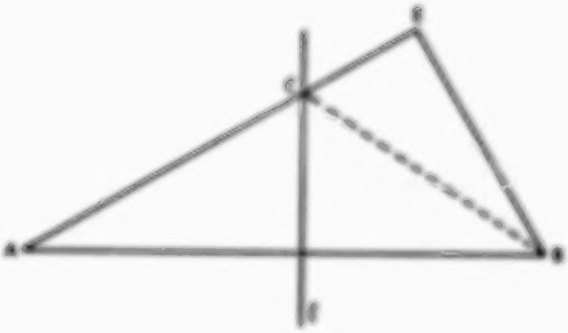
2. Suppose, as shown in the diagram, that $AC \perp BC$. Can AB also be perpendicular to BC. Be ready to support your answer.



3. Suppose, as shown in the diagram, that $\ell_1 \perp \underline{a}$ and $\ell_2 \perp \underline{a}$. Can ℓ_1 intersect ℓ_2? Be ready to support your answer. If they do not intersect, how do you describe their relationship?



4. Let A' be the image of A under a reflection in ℓ, as shown in the diagram, and let AA' intersect ℓ in P. What is the image of P under this reflection? You know that a reflection in a line preserves distance. Compare AP with A'P. We see that $\ell \perp \overline{AA'}$ and P is the midpoint of $\overline{AA'}$. We call ℓ the midperpendicular or perpendicular bisector of AA'. Show that every point in ℓ is as far from A as from A'. We can state the result of this exercise as follows: Every point in the midperpendicular of a line segment is as far from one endpoint of the segment as the other.
5. Suppose ℓ is the midperpendicular of AB. Suppose E is in the B-side of ℓ, as shown in the diagram.
   (a) We can show that $EA > EB$ as follows. You are to give a reason for each statement.



(1) A and B are on opposite sides of ℓ.

(2) E and A are on opposite sides of ℓ.

(3) $\overline{EA}$ intersects ℓ in a point, say C, which is between A and E.

(4) $EA = EC + CA$

(5) $EC + CB > EB$

(6) $CB = CA$

(7) $EC + CA > EB$

(8) $EA > EB$

(b) Suppose F is in the A-side of ℓ. Show by an argument like the one in (a) that $FB > FA$.

(c) State in words the proposition that was proved in (a) and (b).

10.15 Using Coordinates for Line Reflections and Point Symmetries.

For our present purpose we use a special coordinate system in which the axes are perpendicular lines. Such special coordinate systems are called rectangular coordinate systems. We shall study reflections in their axes. Let ℓ_x be the line reflection in the x-axis and let ℓ_y be the line reflection in the y-axis. Let P have coordinates (2,3).

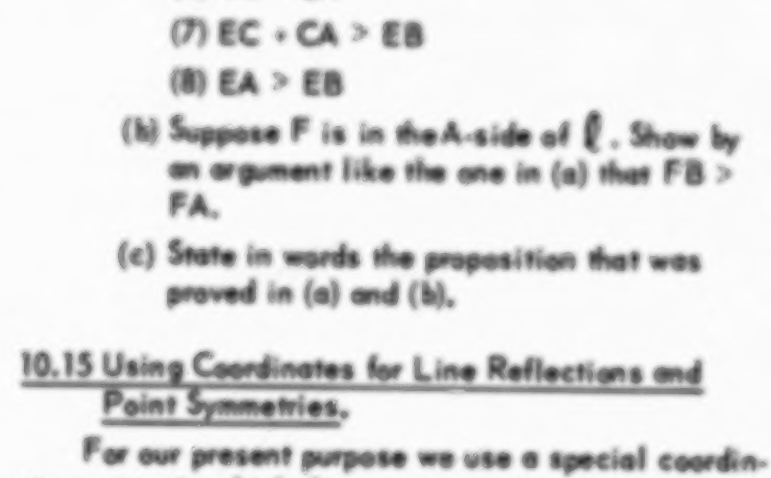


Figure 10.30

If ℓ_x: $P \longrightarrow Q$, what are the coordinates of Q?

If ℓ_y: $P \longrightarrow R$, what are the coordinates of R?

If ℓ_y: $Q \longrightarrow S$, what are the coordinates of S?

We can form the composite of ℓ_y with ℓ_x by taking the reflection in the x-axis, followed by the reflection in the y-axis. What is the image of P under this composite reflection? Does the image of P change if we reverse the order of the reflections?

Now let us consider the same questions for a point A with coordinates (a, b).

If ℓ_x: $A \longrightarrow B$, what are the coordinates of B?

If ℓ_y: $A \longrightarrow C$, what are the coordinates of C?

If ℓ_y with ℓ_x: $A \longrightarrow D$, what are the coordinates of D.

Do you agree that the rules for ℓ_x and ℓ_y, when given in forms of coordinates of points are as follows:

for ℓ_x: $x \longrightarrow x$, $y \longrightarrow -y$ or $(x,y) \longrightarrow (x,-y)$

for ℓ_y: $x \longrightarrow -y$, $y \longrightarrow y$ or $(x,y) \longrightarrow (-x,y)$

for ℓ_y with ℓ_x: $x \longrightarrow -x$, $y \longrightarrow -y$ or $(x,y) \longrightarrow (-x,-y)$.

You must surely have noted by this time that the composite of ℓ_y with ℓ_x is a point symmetry in the origin of the coordinate system. If we denote this symmetry in 0, the origin, as P_0 we can state the rule of P_0 in terms of coordinates as follows:

$$P_0\colon\ (x,y) \longrightarrow (-x,-y)$$

10.16 Exercises

1. For each of the points with coordinates in a rectangular coordinate system given below find the coordinates of its image

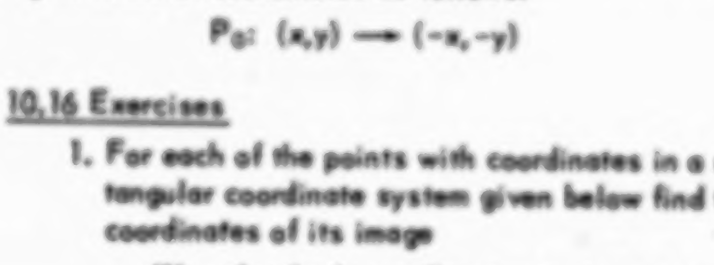
   (1) under the line reflection in the x-axis,

   (2) under the line reflection in the y-axis, and

   (3) under the point symmetry in the origin.

   (a) (3,5) (c) (5,−3) (e) (2,0) (g) (−3,−1)
   (b) (−3,5) (d) (−3,−5) (f) (0,5) (h) (82, −643)

2. Let ℓ be the line that is perpendicular to the x-axis containing the point with coordinates (3,4) in some rectangular coordinate system. Let points have the coordinates listed below. Find the coordinates of the image of each point under a line reflection in ℓ.



   (a) (1,4) (c) (3,2) (e) (0,0) (g) (8,−3)
   (b) (0,3) (d) (−3,−1) (f) (10,0) (h) (x,y)

3. Let m be the line that is perpendicular to the

y-axis of a rectangular coordinate system containing the point with coordinates (3,4). Find the coordinates of and the image of each point in Exercise 2 under a line reflection in m.

4. Find the coordinates of the image of each point in Exercise 2 under a point symmetry in the origin O.

5. Let A and B have rectangular coordinates (1,5) and (3,1) respectively.
   (a) Let ℓ_x: $A \to A'$ and $B \to B'$. Find the coordinates of A' and B'.
   (b) Find the coordinates of the midpoint M of $\overline{AB}$ and let ℓ_x: $M \to M'$. Find the coordinates of M'.
   (c) Show that M' is the midpoint of $\overline{A'B'}$.
   (d) State a proposition suggested by the results of this exercise.

6. Show that the line reflection in the x-axis preserves midpoints. You might wish to work with points A and B having coordinates (2a, 2b) and (2c,2d).

7. Show that the point symmetry in the origin O preserves midpoints.

8. (a) Determine whether the points with coordinates (1,3), (4,1), (10,−3) are on the same line.
   (b) Find the coordinates of the images of the three points in (a) under the line reflection in the x-axis and determine whether or not the images are on a line.
   (c) State in words what the results of this exercise seem to indicate.

9. Using the three points in Exercise 8 show that their images under a point symmetry in the origin are on a line.

10.17 What is an Angle?

No doubt the word "angle" has some meaning for you. However, you may find it quite difficult to describe it precisely. To see just how difficult, you might try to explain what an angle is to a youngster in the first or second grade. A particularly difficult task would be to describe it without diagrams.

(To see how important angles are in everyday thinking, one can look up the word angle and related words in the dictionary. You will be asked to do this in an exercise.)

You probably would say that the diagram in Figure 10.31 represents an angle. But is the entire angle shown in the diagram? Are the rays $\overrightarrow{OA}$ and $\overrightarrow{OB}$ part of the angle? Is the fact that $\overrightarrow{OA}$ and $\overrightarrow{OB}$ have a common endpoint significant? Are the points between A and B part of the angle? These are some of the questions that must be answered in giving a precise mathematical meaning to the word "angle".

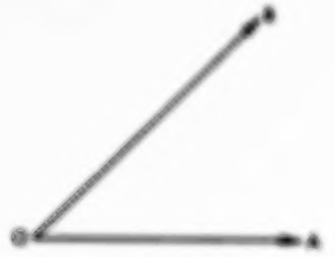


Figure 10.31

After carefully reading the following you should be able to answer all of them.

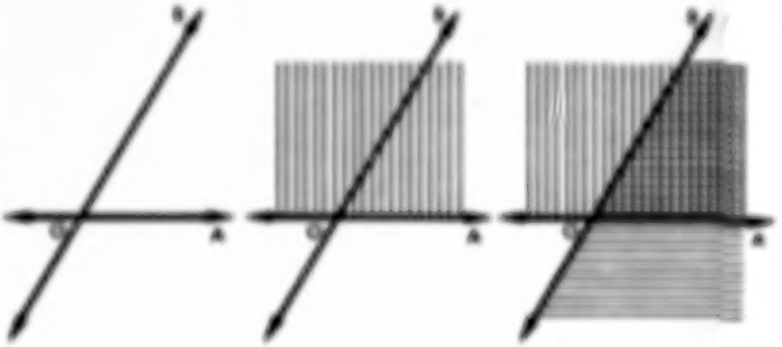


Figure 10.32

Let us start with two lines intersecting at O, as shown in Figure 10.32. We name them $\overleftrightarrow{OA}$ and $\overleftrightarrow{OB}$. With these lines given we shall show in stages how the angle emerges. First we take the halfplane of $\overleftrightarrow{OA}$ that contains B. It is indicated by vertical shading lines. Then we take the halfplane of $\overleftrightarrow{OB}$ that contains A. It is indicated by horizontal shading lines. The region that is cross-hatched is the angle. It is the intersection of the two halfplanes. It is named ∠AOB. Each point used in the name signifies something. O is the point of intersection of the two lines. It is called the vertex of the angle. A and B tell us which halfplane to take. $\overrightarrow{OA}$ and $\overrightarrow{OB}$ are the endrays or sides of the angle. There are other rays in the angle. Any ray starting at O and intersecting any interior point of $\overline{AB}$ is called an interior ray of the angle. All points off the angle, not in endrays, are called interior points of the angle and the set of interior points is called the interior of the angle. If $\overleftrightarrow{OA} = \overleftrightarrow{OB}$ and O is between A

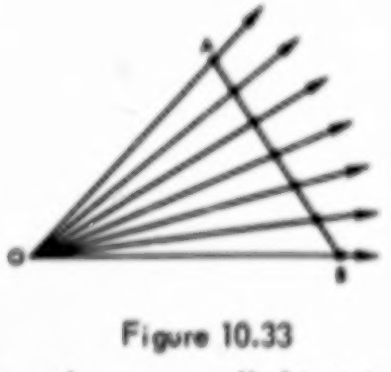


Figure 10.33

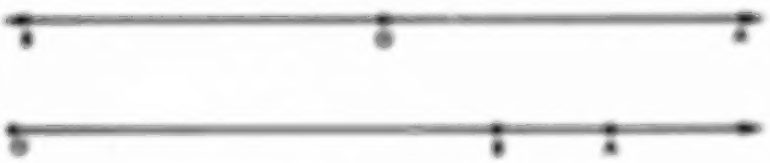


Figure 10.34

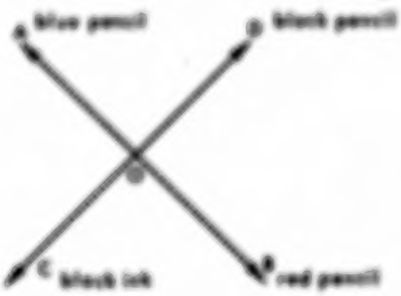
and B, then we cannot build up the angle as described above. Nevertheless we call either halfplane of $\overleftrightarrow{AB}$, with O as vertex, a straight angle. If O is not between A and B, then $\overrightarrow{OA}$ and $\overrightarrow{OB}$ name the same ray. Again we continue to call this an angle, a zero angle.

Does our definition of an angle differ from what you have previously learned about angles?

If so, we ask you to consider the fact that a definition is an agreement among ourselves as to what a word shall mean. Once the agreement is made, however, we must stick with it and with its consequences.

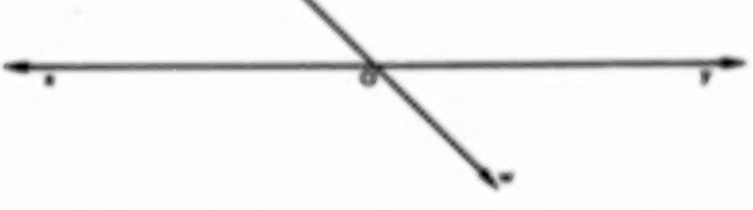
10.18 Exercises

1. Draw two intersecting lines on your paper and label points as in the diagram. Using ordinary

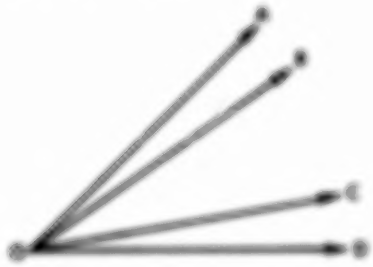


black pencil shade the D-side of $\overleftrightarrow{AB}$ with rays parallel to $\overrightarrow{OD}$, using black ink shade the C-side of $\overleftrightarrow{AB}$ with rays parallel to $\overrightarrow{OC}$. Using red pencil (or any available color) shade the B-side of $\overleftrightarrow{CD}$ with rays parallel to $\overrightarrow{OB}$. Using the blue pencil (or any other available color) shade the A-side of $\overleftrightarrow{CD}$ with rays parallel to $\overrightarrow{OA}$. You can now describe ∠AOD as the blue-black pencil angle. In similar manner describe ∠BOD, ∠AOC, ∠BOC.

2. Using the diagram shown, name:
   (a) two straight angles.
   (b) four zero angles.
   (c) four other angles.



3. Using the diagram shown, describe as a single angle, if possible:
   (a) ∠AOB ∪ ∠BOC
   (b) ∠AOC ∩ ∠COB
   (c) ∠AOC ∪ ∠BOD
   (d) ∠AOC ∩ ∠COD



4. There are ten angles in the diagram of Exercise 3. Four of them are zero angles. Name the other six.

5. You may have noticed that there are many resemblances between an angle and a segment. For each sentence below about segments write one that resembles it and is about angles.
   (a) A segment has two endpoints.
   (b) A segment is a set of points.
   (c) The interior of a segment contains points of a segment other than its endpoints.
   (d) If C and D are interior points of $\overline{AB}$, then every point in $\overline{CD}$ is in $\overline{AB}$.

6. Consult a dictionary to find five uses of angles.

10.19 Measuring an Angle

You have noted above in Exercise 5 a number of resemblances between angles and segments. It should not surprise you that the measurement of angles also resembles the measurement of segments. To measure a segment we use a scaled ruler. To measure an angle we use a scaled protractor. The numbers on a ruler are assigned to points. The numbers on a protractor are assigned to rays (In Figure 10.35 only three rays are shown). Numbers on ordinary rulers start at zero and

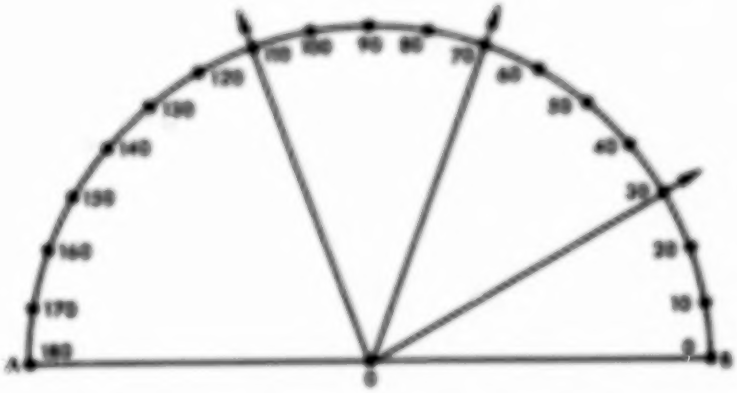


Figure 10.35

go on as far as permitted by the scale unit and the length of the ruler. No matter how large the protractor we are going to use, its numbers start with 0 and end with 180.

As you see, a protractor has the shape of a semicircle. $\overline{AB}$ is the diameter of the protractor and O is its center. In Figure 10.35 the numbers increase in the counter-clockwise direction. However, if we reflect the protractor in the line that is the midperpendicular of $\overline{AB}$, then each number, n, is mapped onto 180-n. In a protractor showing the images of this line reflection, the numbers increase in the clockwise direction (Figure 10.36).

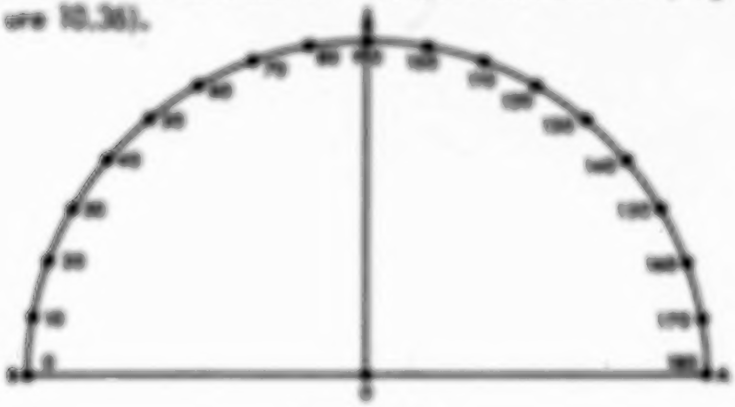


Figure 10.36

In either case the ray which lies in the midperpendicular is assigned 90.

To measure an angle with a protractor we must begin by placing the center O on the vertex of the angle, and each ray of the angle must intersect the edge of the protractor. Perhaps the position of a protractor in measuring ∠ABC could be like that shown in Figure 10.37.

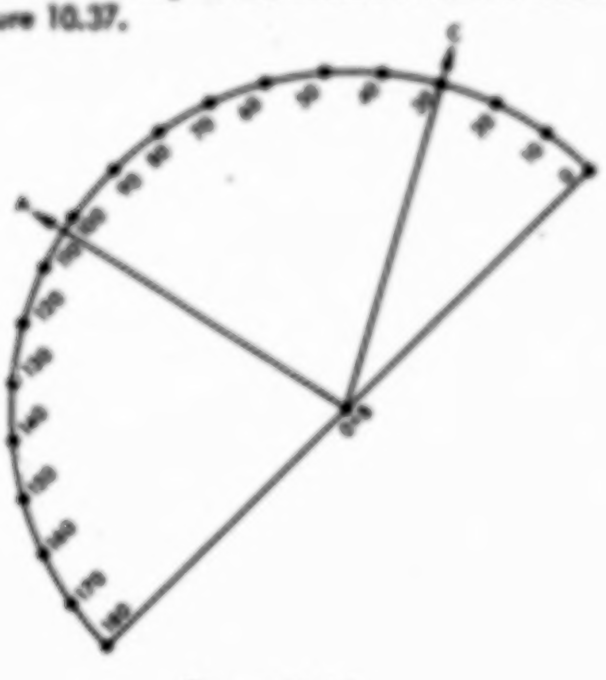
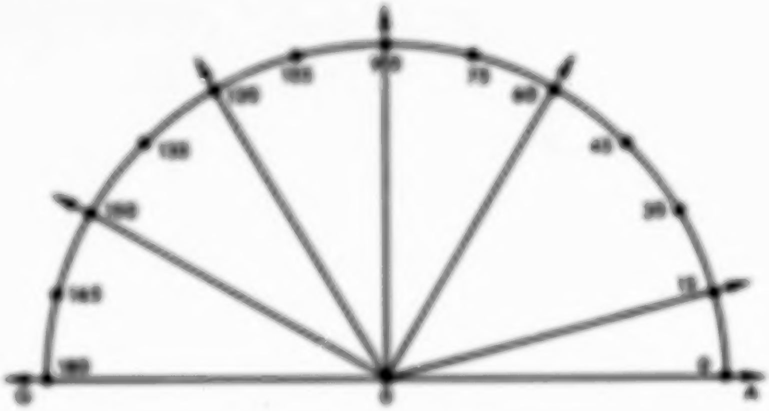


Figure 10.37

In this position the protractor assigns the number 30 to $\overrightarrow{BC}$ and 103 to $\overrightarrow{BA}$. It cannot come to you as a surprise that the measure of ∠ABC is 103-30 or 73. Or if you computed 30-103, you would then take the absolute value of the difference, just as we did in measuring line segments. When the protractor is graduated from 0 to 180 we call the unit of measurement a degree. When we say that the measurement is 73 degrees, or 73°, we are also saying that we used a protractor graduated from 0 to 180. (There are other types of protractors graduated from 0 to other numbers). In measuring a line segment we like to place the ruler so that it assigns 0 to one end, for this considerably simplifies the computation. In measuring an angle we also like to place the protractor so that zero is assigned to an endray, for the same reason.

The abbreviation for "degree measure of ∠ ABC" is m∠ABC.

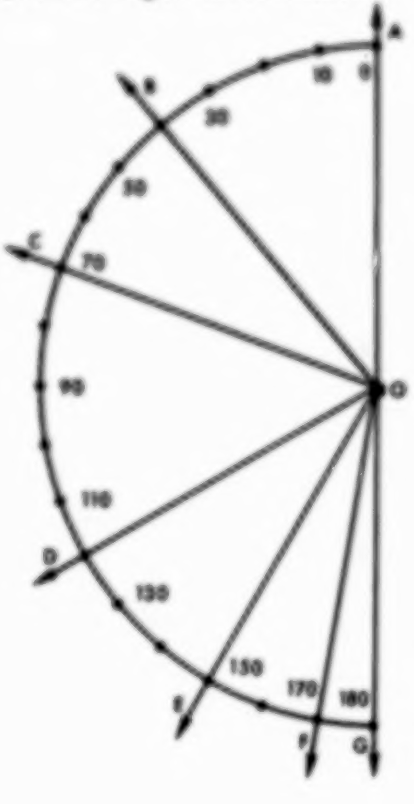
10.20 Exercises



1. Consult the diagram above to find the measure of each angle listed below:

(a) ∠ AOC
(b) ∠ BOC
(c) ∠ COB
(d) ∠ AOF
(e) ∠ BOE
(f) ∠ FOB
(g) ∠ GOC
(h) ∠ EOE
(i) ∠ GOA
(j) ∠ AOG
(k) ∠ AOD
(l) ∠ DOG

2. Using the diagram shown below, find the measure of each angle listed below:



(a) $\angle AOC$ (c) $\angle DOC$ (e) $\angle GOE$

(b) $\angle BOD$ (d) $\angle FOG$ (f) $\angle FOB$

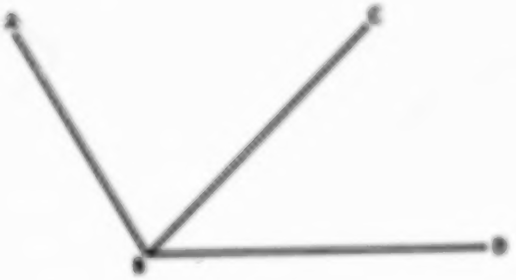
3. Consult the diagram of Exercise 2 to compute each of the following:

(a) $m\angle AOB + m\angle BOC$

(b) $m\angle GOA - m\angle COA$

(c) $2m\angle AOB + 3m\angle OCD$

4. If two angles in a plane have only one ray in common, they are called a pair of adjacent angles. In the diagram determine which pair of angles listed below have only one ray in common.



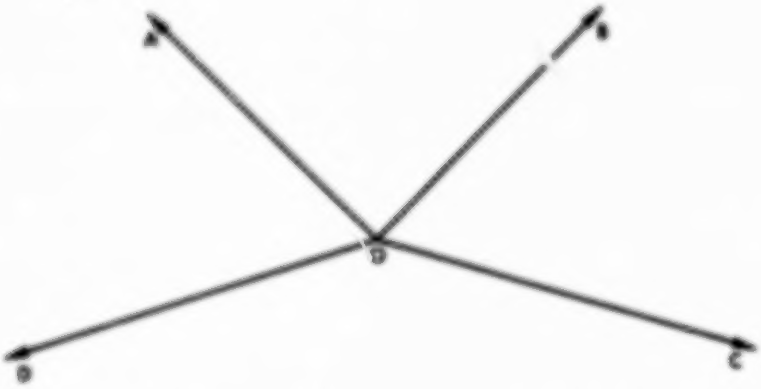
(a) $\angle ABD$ and $\angle CBD$

(b) $\angle ABC$ and $\angle CBD$

(c) $\angle DBA$ and $\angle ABC$

Which is the pair of adjacent angles?

5.



For the given diagram name as many pairs of adjacent angles as you can.

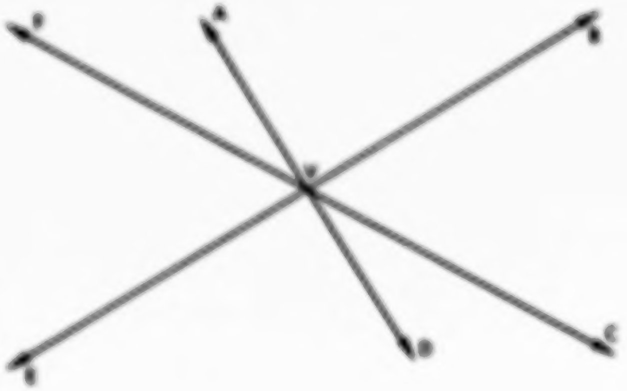
6. Using an illustration show that the sum of the measures of two adjacent angles is not necessarily the measure of an angle.

7. Find the measure of each of the angles listed for the diagram below:

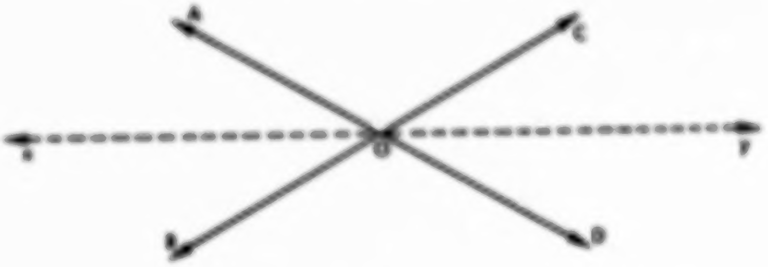
(a) $\angle AVB$ (d) $\angle EVC$ (g) $\angle BVF$

(b) $\angle DVC$ (e) $\angle AVF$ (h) $\angle AVD$

(c) $\angle AVC$ (f) $\angle FVD$



*8. Consider $\angle AOB$, as shown in the diagram and the point symmetry of $\angle AOB$ in vertex O. Under this symmetry the image of endray $\overrightarrow{OA}$ is $\overrightarrow{OD}$, the opposite ray. What is the image of $\overrightarrow{OB}$? What is the image of $\overrightarrow{OX}$, an interior ray of $\angle AOB$? What is the image of $\angle AOB$? The image of an angle under a point symmetry in its vertex is its vertical angle.

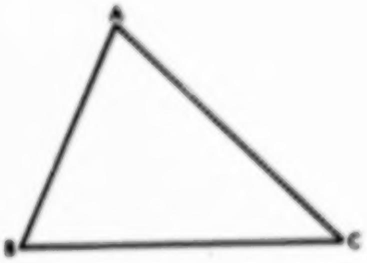


9. (a) In the diagram of Exercise 8, what is the vertical angle of $\angle DOC$?

(b) What is the vertical angle of $\angle AOB$?

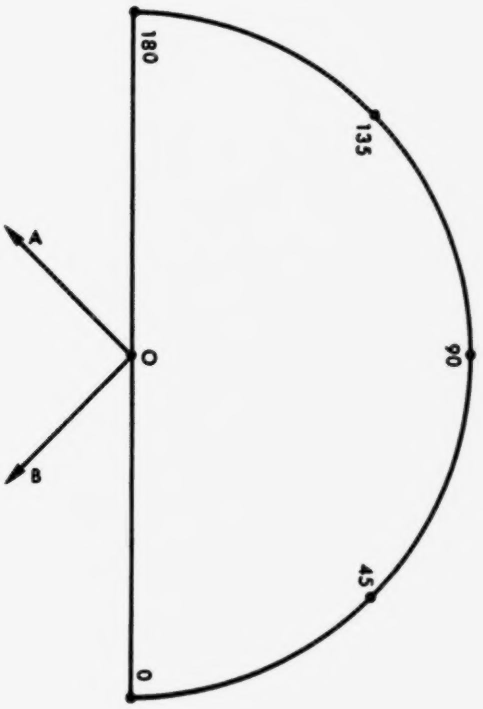
10. Using a protractor show that the measure of an angle is equal to the measure of its vertical angle.

11. $\overline{AB}$ and $\overline{AC}$ are two sides of a triangle. They determine two endrays $\overrightarrow{AB}$ and $\overrightarrow{AC}$ of an angle. In this sense every triangle has three angles. We can name them $\angle A$, $\angle B$ and $\angle C$.



Measure each angle of the triangle and then find the sum of their measures.

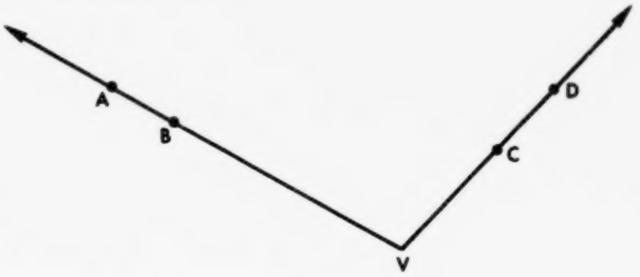
12.



(a) Explain why we cannot use the protractor in the position shown above to measure ∠AOB.

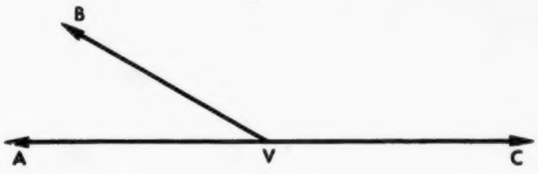
(b) Can the measure of an angle be greater than 180°? Explain your answer.

13. Look at ∠BVC. Now look at ∠AVD. Compare their measures. (Try to answer without the use of a protractor).



14. You know that two perpendicular lines determine four angles. What is the measure of each angle?

15. (a) Measure ∠AVB in the diagram. Using your result, find the measure of ∠BVC.



(b) Suppose the measure of ∠ABV is 70. What is the measure of ∠BVC? Try to answer without using a protractor.

10.21 Boxing The Compass

As you know the marks on a ruler are located by repeated bisections, once we start with inch marks. The first bisection produces a ruler like this:

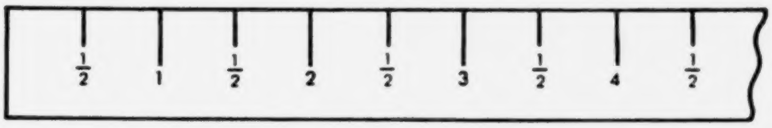


Figure 10.38

A second bisection produces a ruler like this.

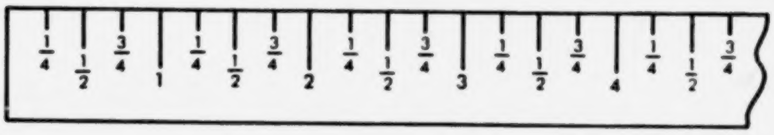
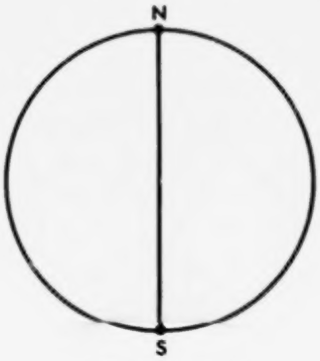


Figure 10.39

Repeated bisections produce eighths, sixteenths and thirtyseconds.

There is an analogous situation for protractors, more accurately for two protractors, placed diameter to diameter to form a circle. It is called boxing the compass, and gives the type of compass used in certain types of marine navigation.

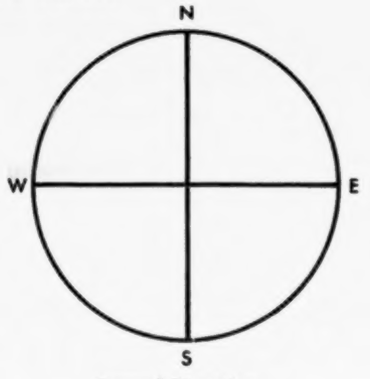
A diameter of either protractor bisects the circle. One end of this diameter is marked N (north) and the other is marked S (south). (Figure 10.40)



First Bisection

Figure 10.40

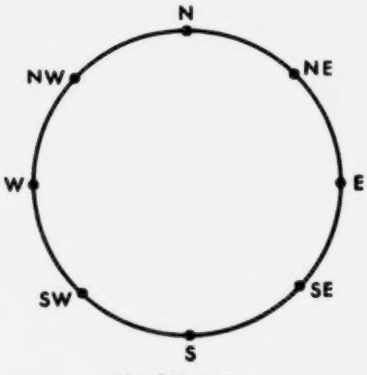
Bisecting each semicircle locates E (east) and W (west). (Figure 10.41)



Second Bisection

Figure 10.41

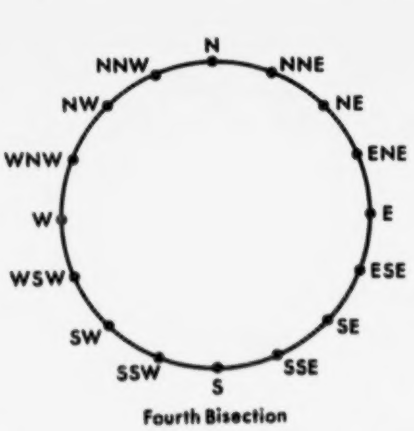
Bisecting each of the four arcs locates NE (north east), SE (south east), SW (south west), and NW (north west). Notice we do not say, "east north." The rule is that "north" takes precedence over "south" because it appeared earlier in the process. Likewise, we say southeast because "south" appears before "east" in the process.



Third Bisection

Figure 10.42

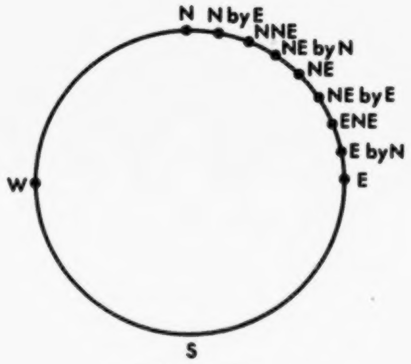
Bisecting each of the eight arcs locates NNE (north northeast), ENE, ESE, SSE, etc. In the designation NNE, N appears before NE because it appeared earlier in the process than NE, and is on the N side of NE. Thus, ENE is on the E side of NE.



Fourth Bisection

Figure 10.43

The fifth bisection completes boxing of the compass. The midpoint of the arc between N and NNE is called N by E (north by east): the one between NNE and NE is called NE by N. Not NNE by S. Why not?



Fifth Bisection

Figure 10.44

Make a complete diagram showing the compass "boxed".

The circle is now subdivided into 32 arcs having the same length. The mariner calls each length a "point". (This point does not mean the point we study in geometry). The terms "halfpoint" and "quarter point" describe still smaller arc lengths. Since there are 8 points to one quarter of a circle, one point corresponds to $11\frac{1}{4}°$. So a change of course of one-quarter point corresponds to a change of approximately 3°.

Thus the kind of "protractor" used in some types of navigation is quite different from the one we have described, with angles measured in "points" from 0 to 16 points east or west of north.

10.22 More About Angles

Draw ray $\overrightarrow{VA}$ on your paper and place your protractor so that $\overrightarrow{VA}$ is assigned zero. In how many

Figure 10.45

possible positions can you hold the protractor? (Were you careful to place the center of the protractor on V?) For each position, draw a ray, starting at V, to which the protractor assigns the number 70. How many such rays can you draw, for each position? How many angles then can you draw having measure 70°, if $\overrightarrow{VA}$ is one of its sides?

Do you agree with this statement?

For each ray, for each halfplane determined by this ray, and for each number x, such that $0 \leq x \leq 180$, there is exactly one angle whose measure is x that has given ray as one side.

This statement is going to be very useful to us in our study of angles. For instance, we can now show that any angle, such as $\angle AVB$, can be divided into two angles that have equal measures. To do this, we

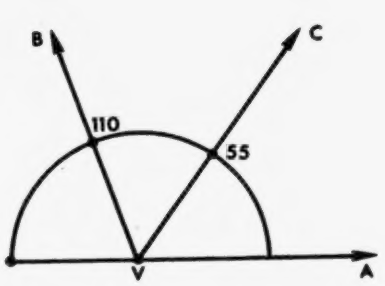


Figure 10.46

place a protractor, in the position shown, see that 110 is assigned to $\overrightarrow{VB}$ and reason that we are looking for the ray that is assigned $\frac{1}{2} \times 110$ or 55. We look for 55 on the protractor and draw $\overrightarrow{VC}$, the ray that is assigned 55. What is $m\angle BVC$? $m\angle CVA$? Have we divided $\angle AVB$ into two angles as claimed? How can we use the statement above to show that an angle has exactly one midray?

In our example $\overrightarrow{VC}$ is called the midray of $\angle AVB$ for obvious reasons; it bisects the angle, and is therefore also called the bisector of $\angle AVB$. Explain why any angle, other than a straight angle, has only one midray.

We pause here to introduce some terms describing angles. If the measure of an angle is 90, it is called a right angle. If the measure of an angle is between 0 and 90, it is called an acute angle. If the measure of an angle is between 90 and 180, it is called an obtuse angle.

10.23 Exercises

1. For each number listed below draw an angle whose measure is that number

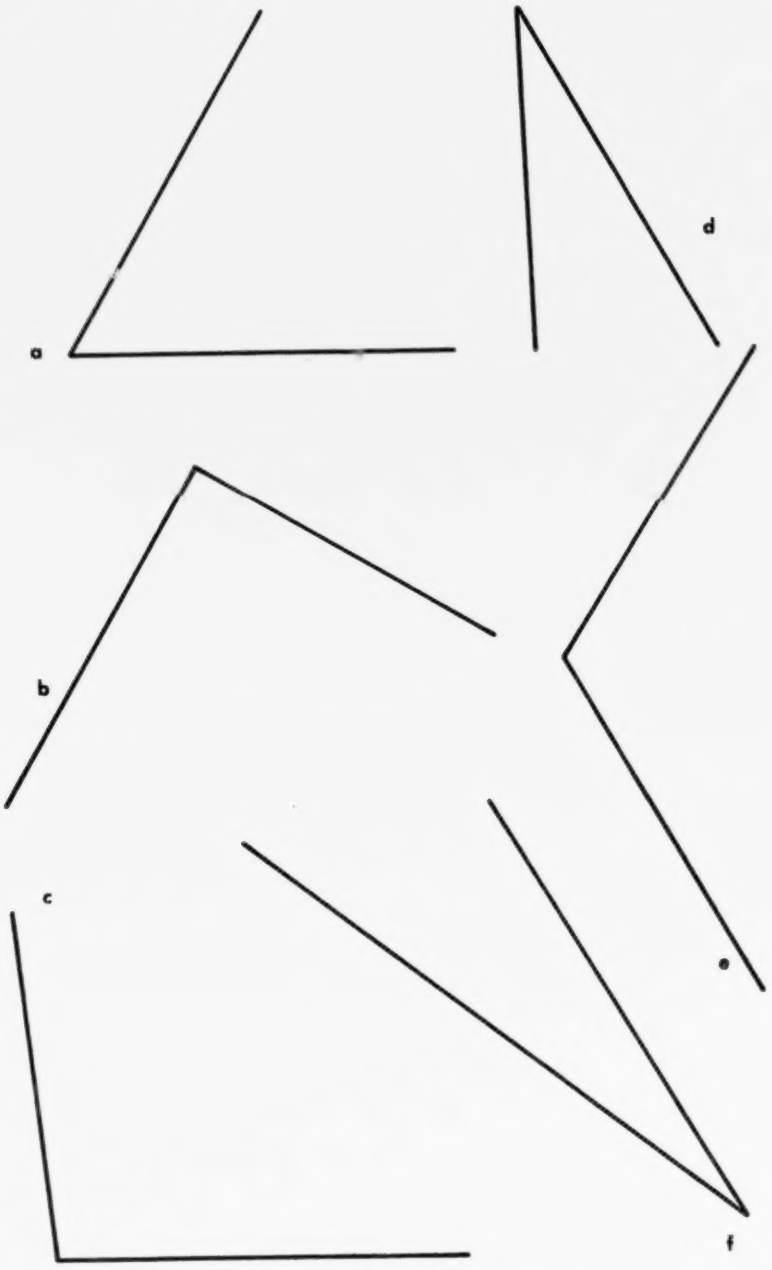
   (a) 35 (b) 135 (c) 18 (d) 90 (e) 180 (f) 0

2. Draw an angle which is:

   (a) a right angle (c) an obtuse angle

   (b) an acute angle

3. This exercise is a test of how well you can estimate the measure of an angle from a diagram. For each of the angles given, estimate the measure, record your estimate, and then use your protractor to check your estimate.



4. This is an exercise to test how well you can draw an angle without protractor when you are told its measure. Draw the angle first, then check with protractor, and record the error, for each of the following measurements:

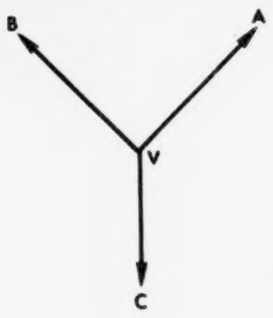
   (a) 45° (c) 150° (e) 60°

   (b) 30° (d) 90° (f) 120°

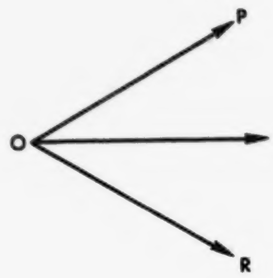
5. How close can you come to drawing the midray of an angle without using a protractor? Try it for these cases: an acute angle, a right angle, an obtuse angle.

6. Try to draw a triangle that has two right angles. If you are not able to do so, explain the failure.

*7. In this exercise, we consider what it means when three rays have the same vertex to say that one is between the other two.



   (a) Look at rays $\overrightarrow{VA}$, $\overrightarrow{VB}$, and $\overrightarrow{VC}$ in the diagram. Would you say that one of them is between the other two? If so, what would you mean?



   (b) Now look at $\overrightarrow{OP}$, $\overrightarrow{OQ}$, $\overrightarrow{OR}$ in the second diagram. Would you say that one of these is between the other two?

   (c) In (a) is $\overrightarrow{VA}$ a ray of $\angle$ BVC? Is $\overrightarrow{VB}$ a ray of $\angle$CVA? Is $\overrightarrow{VC}$ a ray of $\angle$AVB?

   (d) In (b) is $\overrightarrow{OQ}$ a ray of $\angle$ POR?

   (e) Formulate a definition for betweeness for rays.

*8. Draw $\angle$AVB and a ray of this angle that is between $\overrightarrow{VA}$ and $\overrightarrow{VB}$. Name it $\overrightarrow{VC}$. Using a protractor show that $m\angle AVC + m\angle CVB = m\angle AVB$. This result is important enough to have a name. It is the Betweeness-Addition Property of Angles. State it in words. There is also a Betweeness-Addition Property of Segments. State it.

10.24 Angles and Line Reflections

Make a drawing like the one in Figure 10.47, with $\overrightarrow{VM}$ the midray of ∠AVB. (We have an angle of 80°. You can use any angle you like)

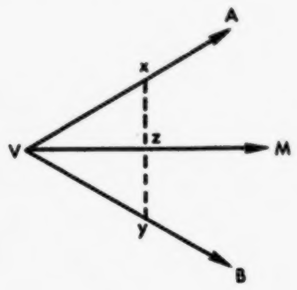


Figure 10.47

If you fold your paper along $\overleftrightarrow{VM}$, do $\overrightarrow{VA}$ and $\overrightarrow{VB}$ fall on each other? Then we may say:

> Each ray of an angle is the image of the other under the line reflection in the midray of the angle.

Suppose X is the point in $\overrightarrow{VA}$ such that VX = 2. Where would you expect to find the image of X under this line reflection? Let X → Y. Then VX = VY. Moreover, the perpendicular to $\overleftrightarrow{VM}$ that contains X must also contain Y. Why? We conclude that $\overleftrightarrow{XY} \perp \overleftrightarrow{VM}$, also if Z is the point in which $\overleftrightarrow{XY}$ intersects $\overrightarrow{VM}$, then XZ = YZ. Why? One more result. In folding your paper, did ∠VXY fall on ∠VYX? Then m∠VXY = m∠VYX. Why?

Let us summarize these results. If $\overrightarrow{VM}$ is the midray of ∠XVY, VX = VY, and $\overleftrightarrow{XY}$ intersects $\overrightarrow{VM}$, then

(1) Under the line reflection in $\overleftrightarrow{VM}$, V → V, X → Y, Z → Z. Since a line reflection is an isometry, VX = VY, XZ = YZ. Also $\overleftrightarrow{XY} \perp \overleftrightarrow{VM}$.

(2) m∠VXZ = m∠VYZ.

The second fact rates attention because it is a special case of a more general statement which we are now ready to understand. It applies to all isometries, of which line reflections are only one kind.

> Under any isometry the measure of an angle is the same as the measure of its image angle.

We shall pursue this further in the next section. Meanwhile, we apply our results to a special type of triangle. If at least two sides of a triangle have the same length it is called an isosceles triangle. These two sides are called the legs of the isosceles triangles the third side is called its base. The angles of the triangle having vertices at the ends of the base are called base angles, the third angle is called the vertex angle. Let △ABC be an isosceles triangle, with AB = AC, and let the midray of the vertex angle intersect the base in point M (Figure 10.48). Then under the line reflection in $\overleftrightarrow{AM}$, A   A, M   M, B   C. By our previous results we conclude:

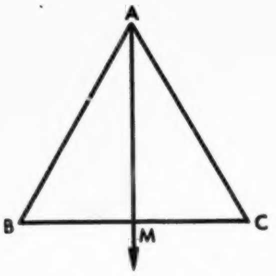


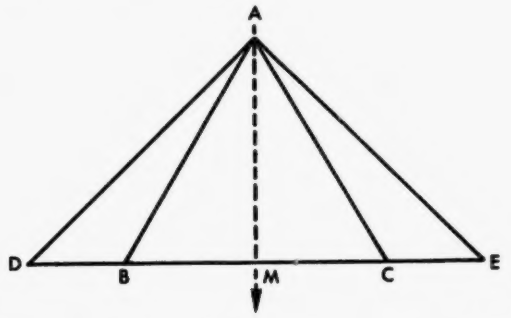
Figure 10.48

(1) The base angles of an isosceles triangle have the same measure.

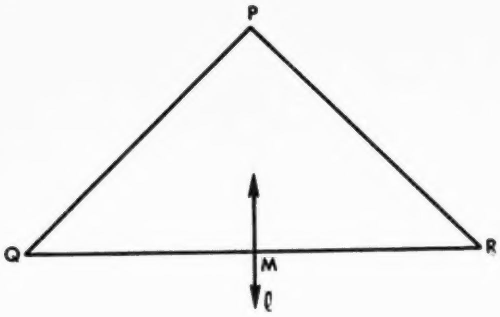
(2) The midray of the vertex angle of an isosceles triangle lies in the midperpendicular of the base.

10.25 Exercises

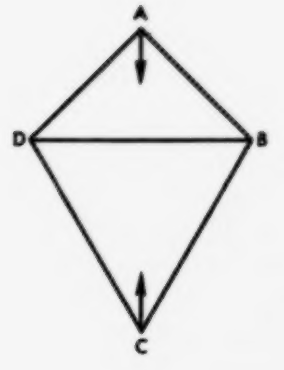
1. Suppose in D, B, C, E are on a line as shown and A is not. If AB = AC, show by an argument that m∠ABD = m∠ACE.



2. For the figure in Exercise 1 add the information that BD = CE. Using the line reflection ℓ in $\overrightarrow{AM}$, the midray of ∠BAC, explain why each of the following is true or false:
   (a) $\overleftrightarrow{AM}$ is the midperpendicular of $\overline{DE}$.
   (b) ℓ : E → D and ℓ : D → E and DM = EM.
   (c) ℓ : $\overline{AD}$ → AE and AD = AE.
   (d) ℓ : $\overrightarrow{AD}$ → $\overrightarrow{AE}$ and $\overrightarrow{AB}$ → $\overrightarrow{AC}$.
   (e) m∠DAB = m∠EAC.
3. Suppose PQ = PR and QM = MR as shown. Let ℓ be the midperpendicular of $\overline{QR}$. Do you think that ℓ contains P? Support your answer with an argument.



4. In the diagram AD = AB and DC = CB:
   (a) What kind of triangle is ABD? CBD?

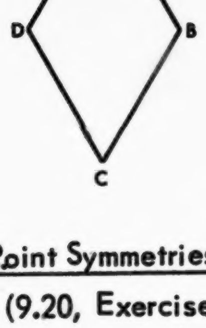


(b) How is the midray of $\angle A$ related to $\overline{BD}$? How is the midray of $\angle C$ related to $\overline{BD}$?

(c) How many midperpendiculars of $\overline{DB}$ are there?

(d) The figure ABCD has the shape of a kite, so we call it a kite. You see that it can be mapped into itself by a line reflection in $\overleftrightarrow{AC}$. List five pairs of angles in the kite for which the angles in each pair have the same measure. Assume that $\overline{AC}$ and $\overline{BD}$, the diagonals, may be inside of these angles.

5. In the diagram the four sides $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{DA}$ have the same length. It is a kind of "double kite". Show that its diagonals bisect each other and lie in perpendicular lines.



10.26 Angles and Point Symmetries.

In an exercise (9.20, Exercise 8) we noted that the image of an angle under a point symmetry in its vertex is its vertical angle. It quickly follows that the measure of an angle is equal to that of its vertical angle. This is a valid conclusion. Nonetheless, let us explore the situation a little more, partly to review some basic notions and partly to illustrate a proof which resembles many that will follow.

Suppose $\angle ABC$ is a given angle (Figure 10.49). If B is the midpoint of $\overline{AA'}$ and also $\overline{CC'}$, then $\angle A'BC'$ is the image of $\angle ABC$ under a point symmetry in B. We can easily locate A′ and C′ by using a compass as divider with B as center. Now look at the quadrilateral ACA′C′. Its diagonals bisect each other. Then what kind of quadrilateral is ACA′C′? How does your answer lead to the conclusion that $CA = C'A'$?

Let us review three facts: (1) $AB = A'B$, (2) $CB = C'B$, (3) $CA = C'A'$. Do not these three facts show that the mapping which maps $A \longrightarrow A'$, $B \longrightarrow B$, $C \longrightarrow C'$ is an isometry? We conclude that $m\angle ABC = m\angle A'BC'$. (Remember that an isometry preserves angle measure.) In this example we reviewed the basic notion of an isometry and we have seen how to use some properties of parallelograms in a proof.

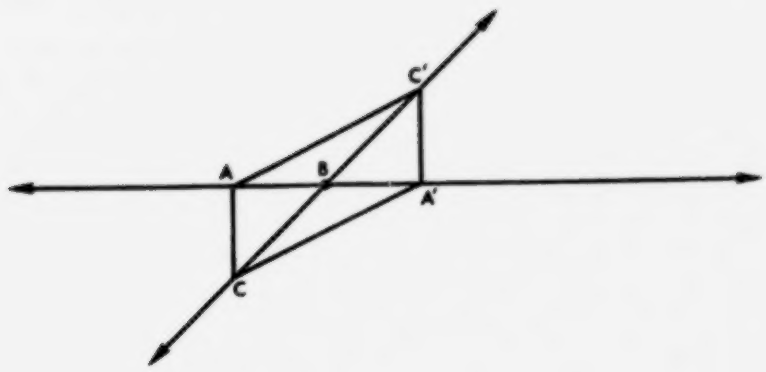


Figure 10.49

Suppose the center of a point symmetry is not the vertex of an angle. In each of Figure 10.50 and 10.51, the image of $\angle ABC$ is $\angle A'B'C'$ under a point symmetry in 0, a point which is not the vertex B. Verify in each case that 0 is the midpoint of $\overline{AA'}$, $\overline{BB'}$, and $\overline{CC'}$. This should assure you that we do indeed have a point symmetry in 0.

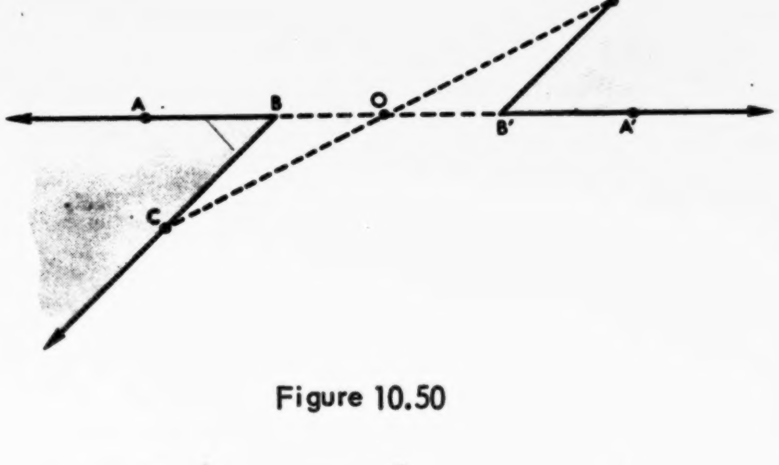


Figure 10.50

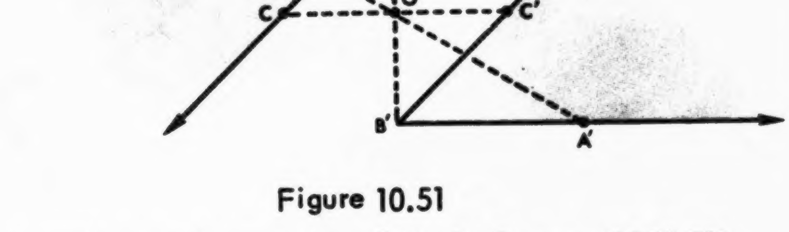
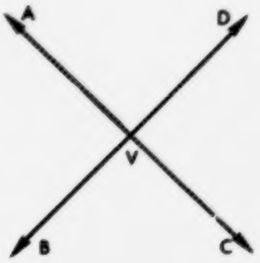


Figure 10.51

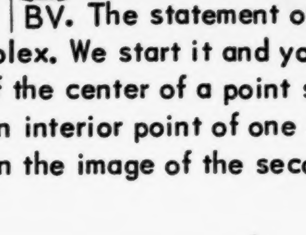
In each case the mapping of (A, B, C) onto (A′ B′ C′) can be shown to be an isometry, that is $AB = A'B'$, $BC = B'C'$ and $CA = C'A'$. Find two parallelograms in Figure 10.50 that help to show why $AC = A'C'$ and $BC = B'C'$. Try to figure out why $AB = A'B'$. In Figure 10.52, we can find three parallelograms that help in proving that the mapping is an isometry. Name the three parallelograms.

10.27 Exercises

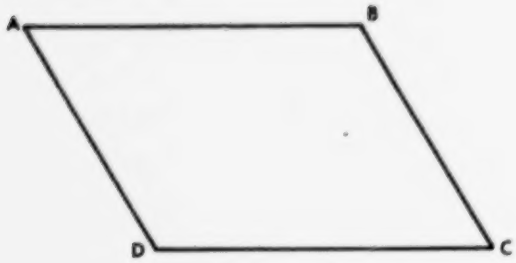
1. Allow yourself the use of a protractor to measure only one of the four angles, $\angle AVB$, $\angle BVC$ $\angle CVD$, $\angle DVA$ and then tell the measures of the other three.



2. Draw a diagram showing the image of $\angle ABC$ under a point symmetry in 0 for each of the following cases.
   (a) 0 is a point in $\overrightarrow{BA}$, not B.
   (b) 0 is a point in $\overrightarrow{BC}$, not B.
   (c) 0 is an interior point of $\angle ABC$.
   (d) 0 is an exteriro point of $\angle ABC$.
3. Copy a figure like the one shown below. Be sure to take 0 as the midpoint of $\overline{VA}$. Draw the image of $\angle AVB$ under a point symmetry in 0. Under this reflection what is the image of V? What is the vertex of the image angle? Show that $\overleftrightarrow{AB'} \parallel \overleftrightarrow{BV}$. The statement of this result is quite complex. We start it and you are to complete it: If the center of a point symmetry of an angle is an interior point of one side of the angle, then the image of the second side .....



4. Draw an angle and its midray, and take any point, not the vertex, of its midray. Draw the image of the angle under a point symmetry in the midray point. You should note that the angle and its image determine a quadrilateral. List some of the properties of this quadrilateral that you can find.
5. Repeat the instructions in Exercise 4 with the modification that the center of symmetry is an interior point of the angle, not in the midray.
6. Suppose ABCD is a parallelogram. Is there a point symmetry under which D→B, A→C? What is its center? How do your answers help to show that each angle of a parallelogram has the same measure as that of the opposite angle?



10.28 Angles and Translations

Let $\angle AVB$ be mapped by a translation such that the image of V is V'.

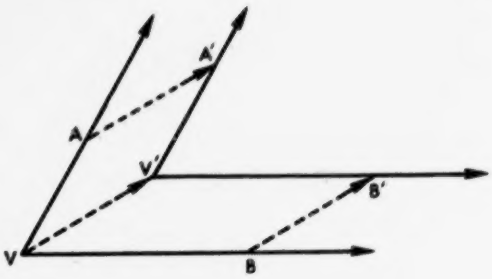
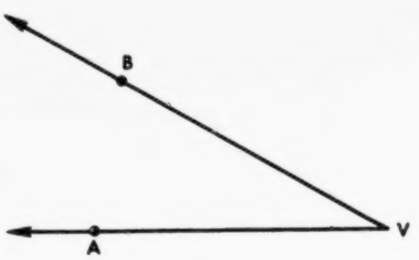


Figure 10.52

Let the images of A and B be A' and B' under this translation. Since a translation is an isometry, and we have agreed that isometries preserve angle measures, it follows that $m\angle A'V'B' = m\angle AVB$. Additional results relating angles and translations are explored in the following exercises.

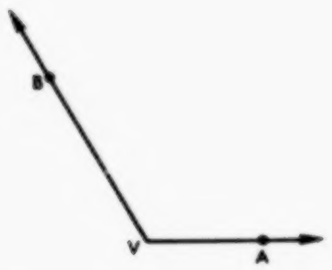
10.29 Exercises

*1. Copy $\angle AVB$ and then show a translation of $\angle AVB$ by a drawing that maps V onto A. Let the translation map A onto A' and B onto B'. Under this translation what are the images of $\overrightarrow{VA}$, $\overrightarrow{VB}$, $\angle AVB$?



We call the pair of angles AVB and A'AB "F angles" because they form an F figure.

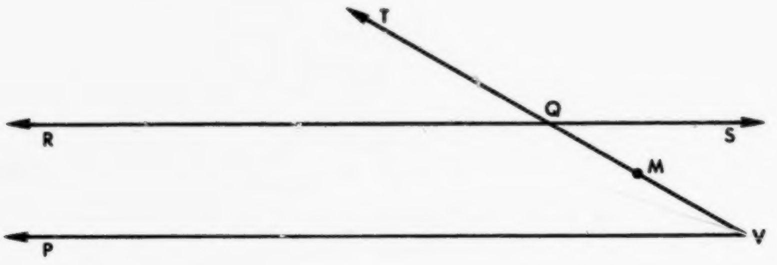
2. (a) Repeat the instructions in Exercise 1 for the translation that maps A onto V.
   (b) Repeat again for the translation that maps V onto B.
3. Let T_1 be the translation that maps A onto V and T_2 the translation that maps V onto B.
   (a) Make a drawing for $T_2 \circ T_1$.



(b) Make a drawing for the composite of T_1 with T_2.

(c) Are the images of $\angle AVB$ under both composites the same? Are the drawings the same?

4. In the diagram below $\overleftrightarrow{RS} \parallel \overleftrightarrow{PV}$ and M is the midpoint of $\overline{QV}$.



(a) Describe a mapping under which the image of $\angle PVQ$ is $\angle RQT$.

(b) Describe a mapping under which the image of $\angle PVQ$ is $\angle VQS$.

(c) Describe a mapping under which the image of $\angle RQT$ is $\angle SQT$. Is this mapping an isometry?

(d) Describe a mapping under which the image of $\angle RQT$ is $\angle SQM$.

(e) Under what composite mapping is $\angle SQM$ the image of $\angle PVQ$, if a translation is first in the composite?

(f) Compare the measures of $\angle PVQ$ and $\angle SQV$.

We call angles PVA and SQV "Z angles" because they form a Z figure.

10.30 Sum of Measures of the Angles of a Triangle.

No doubt you have measured the three angles of a triangle and have found the sum of their measures to be approximately 180. Let us see how isometries can be used to prove this fact.

Figure 10.53 shows an image for each angle of $\triangle ABC$ under different mappings.

First consider the translation that maps A onto C. This translation maps C onto R and B onto S. What are the images of $\overrightarrow{AB}$ and $\overrightarrow{AC}$ under this translation? Do you see that this translation maps $\angle CAB$ onto $\angle RCS$?

Examine the translation that maps B onto C. Under this translation what is the image of $\overrightarrow{BA}$? of $\angle ABC$?

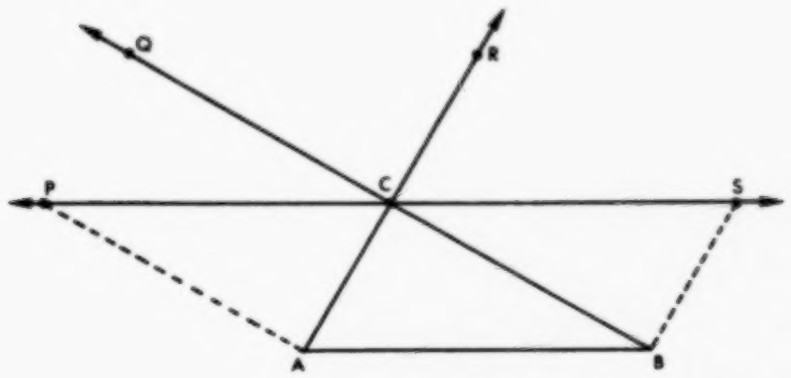


Figure 10.53

The third mapping is a point symmetry in C. Under this mapping what is the image of $\angle ACB$?

As a result of these mappings, all isometries, we see:

(1) $m\angle CAB = m\angle RCS$,
(2) $m\angle ABC = m\angle PCQ$,
(3) $m\angle BCA = m\angle QCR$,

If the sum of the measures of the image angles is 180, then we can safely conclude that the sum of the measures of the angles of the triangle must also be 180.

Do you think the first sum is 180? Why? In answering this question remember that no statement was made concerning whether $\overrightarrow{CS}$ and $\overrightarrow{CP}$ were on one line. Are they? Why?

One can prove the above result by using other isometries, and you may find it interesting (in exercises) to find your own.

There are many immediate results following from the triangle angle measure sum. For instance we can now show: If a triangle has a right angle then the sum of the measures of the other two angles is 90. The proof can be presented in a step by step argument as follows:

1. Let $\triangle ABC$ have a right angle at C.
2. $m\angle A + m\angle B + m\angle C = 180$
3. $m\angle c = 90$
4. $m\angle A + m\angle B = 90$

We can give a valid reason for each of these statements. The reasons, numbered to let you see which reason applies to each statement, are as follows:

1. This information is given in the statement we are trying to prove.
2. We have proved this already. Let us call it the Triangle Angle Sum Property.
3. The measure of a right angle is 90.
4. The cancellation law for addition.

Here is another immediate result with its proof: The sum of the measures of the angles of a quadrilateral is 360.

Figure 10.54 will help you follow the argument.

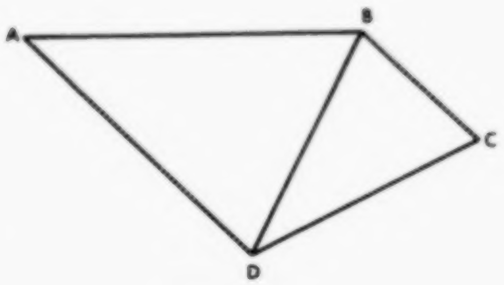


Figure 10.54

We ask you to assume that $\overrightarrow{BD}$ is an interior ray of $\angle ABC$ and $\overrightarrow{DB}$ is an interior ray of $\angle ADC$.

1. $m\angle A + m\angle ABD + m\angle BDA = 180$
2. $m\angle C + m\angle DBC + m\angle BDC = 180$
3. $m\angle ABD + m\angle DBC = m\angle ABC$ or $m\angle B$
4. $m\angle BDA + m\angle BDC = m\angle CDA$ or $m\angle D$
5. $m\angle A + m\angle B + m\angle C + m\angle D = 360$

The reasons for (1) and (2) are the Triangle Angle Sum Property. Statements (3) and (4) have the same reason: if $\overrightarrow{AP}$ is an interior ray of $\angle BAC$, then $m\angle BAP + m\angle PAC = m\angle BAC$.

The reason for statement (5) is: $180 + 180 = 360$.

In exercises you will be asked to prove many other statements which follow from the Triangle Angle Sum Property.

10.31 Exercises

1. Find the measure of the third angle of a triangle if you know the measures of the first two to be as follows:

   (a) 80 and 30 (b) 62 and 49 (c) 40 and 129

2. The measures of two angles of a triangle are the same. What is their measure if the measure of the third angle is:

   (a) 80? (b) 20? (c) 68? (d) 41?

3. What is the measure of each angle of a triangle whose angles all have the same measure?

4. The measures of two angles of a triangle have the ratio 3:5. What are their measures if the third angle has a measure of:

   (a) 100? (b) 68? (c) 30?

5. What is the measure of an angle of a quadrilateral if the measures of the other three angles are:

   (a) 120, 80, 62?

   (b) 100, 62, 62?

   (c) 168, 72, 48?

6. Show that if three angles of a quadrilateral are right angles then the fourth angle must also be a right angle.

7. Let ABCD be a parallelogram. Show that $m\angle A + m\angle B = 180$ and $m\angle C + m\angle D = 180$.

8. Give an argument for each of the following statements. It need not be a step by step argument.

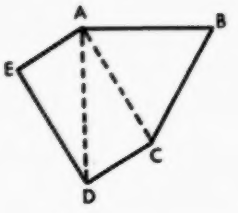
   (a) Two angles of a triangle cannot both be obtuse.

   (b) If a triangle is isosceles then its base angles are acute angles.

9. Prove each of the following. If convenient, use a step by step argument.

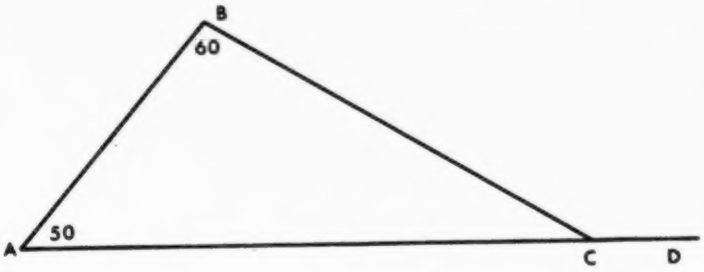
   (a) If in $\triangle ABC$, $AB = BC = CA$, then $m\angle A = 60$.

   (b) The figure below has 5 sides and is called a pentagon. Assume that $\overrightarrow{AD}$, $\overrightarrow{AC}$ are interior rays of $\angle EAB$, and that $\overrightarrow{DA}$ is an interior ray of $\angle EDC$ and $\overrightarrow{CA}$ is an interior ray of $\angle DCB$. Show that the sum of the measures of the angles of ABCDE is 540.



   (c) Assume in (b) that the measures of the angles in ABCDE are the same. Show that each measure is 108.

10. *(a) Using the data indicated below find* $m\angle BCD$



   (b) Suppose $m\angle A = 52$, $m\angle B = 65$. Again find $m\angle BCD$.

   (c) Do the results in (a) and (b) suggest a relationship between $m\angle BCD$ and $m\angle A + m\angle B$?

   (d) Show for all measures $\angle A$ and $\angle B$ that $m\angle BCD = m\angle A + m\angle B$.

11. Find, for the diagram below

   (a) $m\angle ADC$.

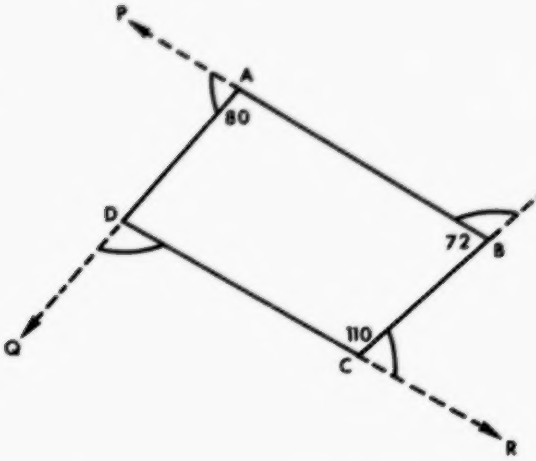
   (b) The measures of the angles, in which arcs are drawn.

(c) The sum of the measures in (b).

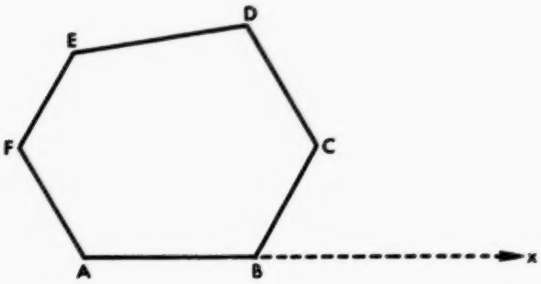
(d) Take another set of measures for the three angles of quadrilateral ABCD and find the sum of the "arc" angles for your new measures.

(e) Do your results in (c) and (d) indicate a pattern? Complete and prove the following statement:

$m\angle BAD + m\angle QCD + m\angle RCB + m\angle SBA = ?$ when ABCD is a quadrilateral.



12. A figure such as ABCDEF has six sides and is called a hexagon.



(a) Find the sum of the measures of its angles.

(b) Let X be a point in $\overrightarrow{AB}$ as shown. It is called an exterior angle of the hexagon. Find the sum of the measures of its exterior angles, one taken at each vertex.

(c) If the angles of a hexagon have the same measure, what is the measure of each angle, and what is the measure of one exterior angle?

13. Repeat Exercise 12 for a figure having 8 sides; 10 sides.

10.32 Summary

This chapter discussed Segments, Angles, and Isometries.

1. The major items relating to segments are the following:

(a) The Line Separation Principle leads to subsets of lines, open halflines and rays, and then to segments.

(b) The distance formula: If x_1 and x_2 are line coordinates of A and B, then $AB = |x_1 - x_2| = |x_2 - x_1|$.

(c) The midpoint formula: If x_1 and x_2 are line coordinates of A and B, then the coordinate of the midpoint of $\overline{AB}$ is $\frac{1}{2}(x_1 + x_2)$.

(d) The Betweeness-Addition Property of Segments: If B is between A and C, then $AB + BC = AC$.

(e) The Triangle Inequality Property: The sum of the lengths of two sides of a triangle is greater than the length of the third.

2. The major items relating to angles are the following:

(a) The Plane Separation Principle leads to open halfplanes, halfplanes, and angles, which are intersections of halfplanes.

(b) The angle measure formula: If r_1 and r_2 are the numbers assigned by a protractor to two sides of an angle, the measure of the angle is $|r_1 - r_2| = |r_2 - r_1|$.

(c) Boxing the compass is accomplished by the repeated bisection of arcs or angles, comparable to the bisection method used in graduating a ruler.

(d) Angles are classified as zero, acute, right, obtuse and straight angles.

(e) The Betweeness-Addition Property of Angles: If $\overrightarrow{VB}$ is between $\overrightarrow{VA}$ and $\overrightarrow{VC}$, then $m\angle AVB + m\angle BVC = m\angle AVC$.

Isometries. The major item is: Isometries preserve angle measures.

(a) Using line symmetries we can show:

(1) An angle is its own image under the line reflection in its midray. This leads to related isosceles triangle properties, and kite properties.

(2) Every point in the midperpendicular of a line segment is as far from one endpoint of the segment as from the other.

(3) The rectangular coordinate formula for the reflection in the x - axis is $(x,y) \longrightarrow (x,-y)$, for the reflection in the y - axis, $(x,y) \longrightarrow (-x,y)$.

(b) Using point symmetries we can show:

(1) The measure of an angle is the same as that of its vertical angle.

(2) The measures of opposite angles of a parallelogram are the same.

(3) The angles in a "Z figure" have the same measure.

(4) The coordinate formula for the point symmetry in the origin of a coordinate system is $(x,y) \rightarrow (-x,-y)$.

(c) Under a translation we can show:

a. The angles in an "F figure have the same measure

b. The coordinate formula for a translation is: $(x,y) \rightarrow (x+p, y+q)$, if the origin is mapped onto (p,q).

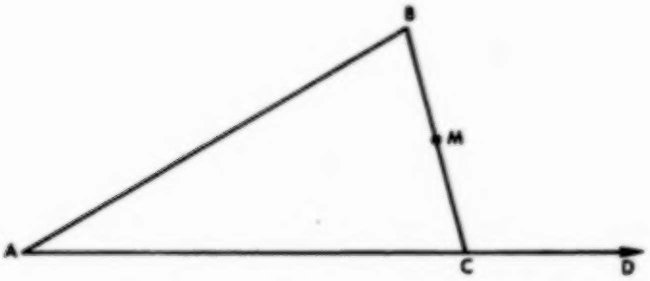
3. Using point symmetries and translations we can show why the sum of the measures of angles of a triangle is 180. This leads to a long list of immediate results.

10.33 Review Exercises

1. Let a mathematical ruler assign −2 to point A and 4 to point B.
   (a) What is AB?
   (b) What number does the ruler assign to the midpoint of $\overline{AB}$?
   (c) C is a point in $\overleftrightarrow{AB}$. If AC + CB = AC what are the possible assignments the ruler can make to C?
   (d) If D is between A and B and AD = 2DB what is the number assigned to D?
   (e) If D is in $\overleftrightarrow{AB}$, not between A and B, and AD = 2DB what is the number assigned to D?
   (f) What numbers may be assigned to point E if AE = 6 and E is in $\overleftrightarrow{AB}$?
2. In Exercise 1 replace −2, the number assigned to A, with −12 and replace 4, the number assigned to B, with −6. Answer the questions in Exercise 1 for these replacements.
3. A protractor assigns 10 to $\overrightarrow{VA}$ and 110 to $\overrightarrow{VB}$:
   (a) What is $m\angle AVB$?
   (b) What number does the protractor assign to the midray of $\angle AVB$?
   (c) The protractor assigns 120 to $\overrightarrow{VD}$. Is $\overrightarrow{VD}$ between $\overrightarrow{VA}$ and $\overrightarrow{VB}$?
   (d) What must be true of x if x is the number assigned to a ray that is between $\overrightarrow{VA}$ and $\overrightarrow{VB}$?
   (e) Suppose $\overrightarrow{VX}$ is a ray of $\angle AVB$, what is $m\angle AVX + m\angle XVB$?
   (f) Suppose $\overrightarrow{VY}$ is a ray of $\angle AVB$ such that $m\angle AVY = 2m\angle YVB$. What number does the protractor assign to $\overrightarrow{VY}$?
4. In Exercise 3 replace 10, the number assigned to $\overrightarrow{VA}$, with 122, and replace 110, the number assigned to $\overrightarrow{VB}$, with 38. Then answer the questions in Exercise 3 for these replacements.
5. Try to draw a triangle such that one of its angles is a right angle and another is an obtuse angle. Explain how you were able to or not able to make the drawing.
6. In a certain rectangular coordinate system A, B, and C have coordinates (−4, 2), (1, −3) and (6, 2) respectively.
   (a) What are the coordinates of A′, B′, C′, the images of A, B, and C, under the line reflection in the x − axis?
   (b) Are A, B, C collinear? Are A′, B′, C′ collinear?
   (c) Compare AB with A′ B′ make the comparison without finding the numbers AB and A′ B′ and justify your answer.
   (d) Compare $m\angle ABC$ with $m\angle A'B'C'$ after measuring each angle with a protractor. Can you make the comparison without using a protractor? Justify your answer.
7. Answer the questions in Exercise 6 if A′, B′, and C′ are the images of A, B, and C under the line reflection in the y − axis.
8. Answer the questions in Exercise 6 if A′, B′, and C′ are the images of A, B, and C under the point symmetry in the origin of the coordinate system.
9. Answer the questions in Exercise 6 A′, B′, and C′ are the images of A, B, and C under the point symmetry in P(1,2).
10. Answer the questions in Exercise 6 if A′, B′, and C′ are the images of A, B, and C under the line reflection in the line perpendicular to the x − axis and containing P(1, 2).
11. Consider the coordinate rule by which (x,y) is mapped onto (y,x) in a rectangular coordinate system.
    (a) Under this mapping what are the coordinates of the images of (2,0), (0,4), (−1,2), (3,3), (−5,−2), (0,0)?
    (b) Make a graph of the points in (a) and their images.
    (c) Is this mapping a line translation, a point symmetry, a translation, or none of these? If it is, describe it, giving domain, range and the rule for its inverse mapping.
    (d) What is the composition of this mapping with itself?
12. Consider the coordinate rule in a rectangular coordinate system by which $(x,y) \rightarrow (-y,-x)$. Answer the questions in Exercise 11 for this mapping.

13. Is the mapping with coordinate rule (x,y) $(2x,2y)$ in a rectangular coordinate system an isometry?

14. Let M be the midpoint of $\overline{BC}$ in $\triangle ABC$. Using a point symmetry in M and a translation show how to prove that $m\angle A + \angle B + m\angle C = 180$.

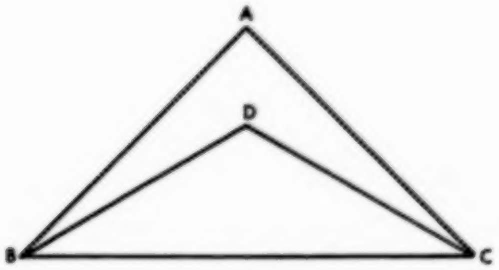


15. Find the measure of an angle of an n-sided figure, where angles have the same measure, and n has the value given below.

(a) $n = 6$
(b) $n = 3$
(c) $n = 8$
(d) $n = 12$
(e) $n = 20$

16. Find the measure of an exterior angle of each n-sided figure in Exercise 15.

17. In the figure below $AB = AC$, and $DB = DC$. Using a line reflection, prove $m\angle DAB = m\angle DAC$.



CHAPTER 11

ELEMENTARY NUMBER THEORY

11.1 (N, +) and (N, ·)

Over the centuries many discoveries have been made concerning properties that various sets of numbers possess. In this chapter we shall concentrate on seeking out properties of certain subsets of the whole numbers. In particular we shall examine the set of natural numbers. (By the natural numbers, N, we mean the whole numbers with zero deleted.) We shall begin by stating certain basic assumptions concerning the natural numbers. Such assumptions, that is statements which we agree to accept as true, are called *axioms*. We shall use these axioms to prove other statements which we call *theorems*. In fact, number theory provides us with a large source of simple and important theorems from which we can begin to learn some of the basic ideas dealing with "proof."

Before stating the first axiom let us recall a problem considered in Chapter 2: [See Exercise 12 on page 35] "Is addition an operation on the set of odd whole numbers?" It is easy to find an example which indicates the answer to this question is "no". Both 3 and 5 are odd whole numbers but their sum, 8, is *not* an odd whole number. Because the set of odd whole numbers is a subset of W we see that addition is not an operation on every subset of W. Thus any statement which asserts that addition is an operation on a subset of W is a non-trivial statement. Our first axiom (A1) states that addition is an operation on N.

A1. (N, +) is an operational system.

Because $3 \epsilon N$ and $5 \epsilon N$ we can conclude, by A1, that $3 + 5 = 8 \epsilon N$. In general A1 states that given any ordered pair of natural numbers we can assign to this pair a unique natural number called their *sum*.

An obvious question to consider next is the following: "Is multiplication an operation on N?" Our second axiom provides the answer to this question.

A2. (N. ·) is an operational system.

Since $3 \epsilon N$ and $5 \epsilon N$ we can conclude by A2 that $3 \cdot 5 = 15 \epsilon N$. In general, A2 states that given any ordered pair of natural numbers we can assign to this pair a unique natural number called their *product*. For example,

$$(3,5) \xrightarrow{\cdot} 15$$

We frequently express the above by the mathematical sentences

$$3 \cdot 5 = 15 \quad \text{or} \quad 3 \times 5 = 15$$

Let us review some of the language used in discussing the operational system (N, ·). In the sentence above 3 is said to be a *factor* of 15. Also, 5 is said to be a factor of 15.

Definition 1: We say that for a and b in N, a is a *factor* of b if and only if there is some natural number c such that $a \cdot c = b$.

Thus 3 is a factor of 15 because there is a natural number, 5, such that $3 \cdot 5 = 15$. 4 is not a factor of 15 because there is no natural number c such that $4 \cdot c = 15$. 5 is a factor of 15 because $5 \cdot 3 = 15$.

Recall that in Chapter 2 you were introduced to the idea of *multiple*. For the mathematical sentence

$$3 \cdot 5 = 15$$

we say that 15 is a multiple of 3 and also that 15 is a multiple of 5.

Definition 2: For a and b in N, b is a *multiple* of a if and only if a is a factor of b.

Thus for the mathematical sentence

$$4 \times 9 = 36$$

we can make the following statements:

4 is a factor of 36

9 is a factor of 36

36 is the product of the factors 4 and 9

36 is a multiple of 4

36 is a multiple of 9

In Chapter 8 we made frequent use of the binary relation "divides" on various sets of numbers. In this chapter we again make use of this relation. In particular, if 4 is a factor of 36 we say that 4 *divides* 36 and we write

$$4 \mid 36$$

Definition 3: We say that for a and b in N, a *divides* b if and only if a is a factor of b. We denote "a divides b" by "a | b".

For the sentence

$$3 \times 4 = 12$$

we can make the following statements:

3 is a factor of 12

3 divides 12

$3 \mid 12$

4 is a factor of 12

$4 \mid 12$

12 is a multiple of 4, etc.

Since 5 is not a factor of 12 we can say that 5 does not divide 12 (sometimes written $5 \nmid 12$).

Because $1 \cdot n = n$ where n is any natural number v see that 1 is a factor of every natural number. Also, every natural number is a multiple of 1.

Question: Can we say that $1 \mid n$ for all n in N? Explain.

You are familiar with the idea that every natural mber has many names. A number such as 12 can e renamed in many ways:

$10 + 2$ $\qquad$ $3 \cdot 4$

$1 \cdot 12$ $\qquad$ $6 \cdot 2$

e shall use the words *product expression* to talk bout names such as "1 · 12" and "3 · 4" that volve multiplication. We say that "1 · 12" and 3 · 4" are product expressions of 12. It is possible have product expressions for 12 with more than vo factors such as:

$1 \cdot 2 \cdot 6$ $\qquad$ $2 \cdot 2 \cdot 3$

$1 \cdot 3 \cdot 4$ $\qquad$ $1 \cdot 2 \cdot 2 \cdot 3$

e see that we can use any of several different product xpressions to represent the number 12.

Question: How many product expressions of 12 are there which contain exactly two factors?

Question: Is 59·509 a product expression for 30031? (the number 30031 will be mentioned later in this chapter in connection with an important theorem).

In this section we have considered some of the basic anguage used in number theory. Again, for a and b in N, a is a *factor* of b if there is some natural number c such hat $a \cdot c = b$. Thus, 7 is a factor of 21 because $7 \cdot 3 = 21$. If a is a factor of b, we say that b is a *multiple* of a. Thus 21 is a multiple of 3. If a is a factor of b, we say that a *divides* b (written $a \mid b$). Thus $7 \mid 21$ and $3 \mid 21$. We say that 21 is the *product* of the actors 7 and 3. Also we say that "7 · 3" is a *product expression* for 21. Note that the words *product expression* are used to talk about names such as "7 · 3" and "1 · 21" and "1 · 3 · 7" that involve finding a product.

11.2 Exercises

1. Explain why the following are, or are not, true:
   (a) $(2 + 3) \in N$
   (b) $(2 \cdot 3) \in N$
   (c) If $a \in W$ and $b \in W$, then $(a + b) \in N$
   (d) If $x \in N$ and $y \in N$, then $(x + y) \in N$
   (e) If $p \in N$ and $q \in W$, then $(p \cdot q) \in N$
   (f) The product of two natural numbers is a natural number.

2. Complete the following sentences:
   (a) If a is a factor of b, then b is a __?__ of a.
   (b) If $x \cdot y = z$, then __?__ is a factor of __?__.
   (c) If $p \cdot q = r$, then __?__ is a multiple of ______.
   (d) If $5 \mid 100$, then 5 is a __?__ of 100.
   (e) If $7 \cdot 8 = 56$, then 56 is called the __?__ of __?__ and __?__.
   (f) If $9 \cdot 7 = 63$, then "9 · 7" is called a __?__ of 63.

3. Determine if the following are or are not true. Explain your answers.
   (a) 3 is a factor of 18
   (b) 7 is a factor of 17
   (c) 3 is a factor of 10101
   (d) 12 is a factor of 96
   (e) 30 is a factor of 510
   (f) 1 is a factor of 3
   (g) 8 is a factor of 8
   (h) 65 is a multiple of 13
   (i) 91 is a multiple of 17
   (j) 5402 is a multiple of 11
   (k) 10 is a factor of 1000 because $10 \cdot 100 = 1000$
   (l) 16 is a factor of 8 because $8 \cdot 2 = 16$

4. Determine if the following are or are not true. Explain your answer.
   (a) $3 \mid 39$
   (b) $17 \mid 91$
   (c) $8 \mid 4$
   (d) $1 \mid 4$
   (e) $13 \mid 65$
   (f) $3 \mid 6$, $3 \mid 12$ and $3 \mid 18$
   (g) $2 \mid n$ where is any even natural number
   (h) $n \mid n$ where n is any natural number
   (i) $n \mid n^2 + 3n$ for all n in N

5. For the following numbers determine all product expressions which contain exactly two factors.
   (a) 6
   (b) 7
   (c) 1
   (d) 12
   (e) 13
   (f) 2
   (g) 3
   (h) 35
   (i) 36
   (j) 37

11.3. Divisibility

In this section we shall consider how sentences dealing with natural numbers can be established as theorems. An example of such a sentence is the following:

If a is an even natural number and b is an even natural number then $a + b$ is an even natural number. This sentence was *assumed* to be true earlier in our text (See, for example, Chapter 4, Exercise 2a, page 76). Our goal now is to *prove* that $a + b$ must be an even natural number whenever a and b are even natural numbers. In order to prove this some additional axioms for $(N, +, \cdot)$ are needed. Rather that just stating those axioms needed to prove the above sentence, we now record a number of additional axioms for $(N, +, \cdot)$ which may be used to prove many other theorems. Note that these axioms summarize properties of $(N, +, \cdot)$ you have already been using.

A3. For all a and b in N, $a + b = b + a$ and $a \cdot b = b \cdot a$.

A4. For all a, b, and c in N,

$a + (b + c) = (a + b) + c$ and $a \cdot (b \cdot c) = (a \cdot b) \cdot c$.

A5. For all a, b, and c in N,

$a \cdot (b + c) = (a \cdot b) + (a \cdot c)$.

A6. For all a in N, $a \cdot 1 = 1 \cdot a = a$.

Question: What familiar names do we give to the axioms A3 – A6?

In addition to these properties of natural numbers, we will make frequent use of a general logical principle that we first stated in Chapter 6. It is the *Replacement Assumption.*

> The mathematical meaning of an expression is not changed if in this expression one name of an object is replaced by another name for the same object.

As an illustration, consider the use of the cancellation property in solving the equation $7 . 2 + x = 46$. Another name for 46 is $(7 . 2 + 38.8)$. Therefore, using the Replacement Assumption, we can write

$$7 . 2 + x = 7 . 2 + 38.8$$

and conclude that $x = 38.8$.

There are two specific ways in which the Replacement Assumption will be used in establishing proofs of sentences about the natural numbers. These are contained in the following theorem.

Theorem A. If a, b, c, and d are natural numbers such that $a = b$ and $c = d$, then

1) $a + b = c + d$
2) $a \cdot b = c \cdot d$.

Proof:

1) Clearly, $a + c = a + c$. Since $c = d$ means that "c" and "d" are two names for the same object, we can replace any "c" by "d" without changing the mathematical meaning of the expression involved. Using this replacement we have $a + c = a + d$. Similarly, since $a = b$ means that "a" and "b" are names for the same object, we can replace any "a" by "b" without changing the mathematical meaning of the expression involved. Therefore, $a + c = b + d$. Note that the two replacements were made for the "a" and "c" to the right of the " = " in $a + c = a + c$.

2) To show that $a \cdot c = b \cdot d$ we proceed in a similar manner. Certainly $a \cdot c = a \cdot c$. Replacing "c" with "d" and "a" with "b" to the right of the " = " we obtain $a \cdot c = b \cdot d$.

Let us now consider how we can prove the sentence about even natural numbers with which we began this section. Before beginning the proof we note that a natural number n is defined to be even if and only if $2 \mid n$. Our proof proceeds as follows.

Since a is an even natural number, we know that $2 \mid a$ or that 2 is a factor of a. By Definition 1 this means that there is a natural number x such that $a = 2 \cdot x$. Similarly, since b is an even natural number, $2 \mid b$ and there is a natural number y such that $b = 2y$. Then, by the first part of the Theorem A just proven, $a + b = 2 \cdot x + 2 \cdot y$. But $2 \cdot x + 2 \cdot y = 2 \cdot (x + y)$ by the Distributive Property, A5. Hence, we may use the Replacement Assumption to obtain $a + b = 2 \cdot (x + y)$. Since $x \in N$ and $y \in N$ then, by A1, $(x + y) \in N$. We see that according to Definition 1 this means that $2 \mid (a + b)$. Hence $a + b$ is an even natural number and the proof is complete.

We can also express the above in the following manner using "parallel columns." That is statements used in the "proof" appear in the left column and justifications of these statements appear in the right column.

Theorem: If $2 \mid a$ and $2 \mid b$, then $2 \mid a + b$ where a and b are natural numbers.

Proof:

1. $2 \mid a$ and $2 \mid b$	1. Given
2. $a = 2x$ and $b = 2y$ where $x, y \in N$	2. Definition 1.
3. $a + b = 2x + 2y$	3. Theorem A
4. $2x + 2y = 2\,(x + y)$	4. A5 (" · " is Distributive over " + ")
5. $a + b = 2 \cdot (x + y)$	5. Replacement Assumption
6. $(x + y) \in N$	6. Statement 2 and A
7. $2 \mid (a + b)$	7. Definition 1 (definition of " | ").

If we call the above "a proof" of the theorem

If $2 \mid a$ and $2 \mid b$, then $2 \mid (a + b)$ (1)

we mean that we have shown that the conditional sentence (1) (i.e., a sentence of the "if p, then q" type) is true *for all values of the variables* a and b. It is possible to generalize sentence (1) to obtain

If $c \mid a$ and $c \mid b$, then $c \mid (a + b)$ where $a, b, c \in N$ (2)

In order to give a proof of (2) one must show that it is true for all natural numbers a, b, and c. (This will be asked for in an exercise.)

Question: Would sentence (2) be proven as a theorem if we proved it true for $c = 3$?

We have settled the question concerning the sum of any two even natural numbers. But what can be said concerning the product of two such numbers? A little experimentation (e.g., $2 \cdot 4 = 8$, $6 \cdot 8 = 48$, etc.) suggests we attempt to prove the following theorem:

If $2 \mid a$ and $2 \mid b$, then $2 \mid a \cdot b$

Our proof might proceed as follows: (See if you can answer each of the "Why?" questions.) Since we are given that $2 \mid a$ and $2 \mid b$ we can state that $a = 2x$ and $b = 2y$ where x and y are natural numbers (Why?). Further $a \cdot b = (2x) \cdot (2y)$ (Why?). But $(2x) \cdot (2y) = 2 \cdot [x \cdot (2y)]$ (Why?) Thus $a \cdot b = 2 \cdot [x \cdot (2y)]$ (Why?) Since the number in the brackets is a natural number (Why?) we conclude that $2 \mid a \cdot b$. (Why?)

If you have been able to justify each of the statements in the above argument then you have a proof of the conditional sentence

If $2 \mid a$ and $2 \mid b$, then $2 \mid a \cdot b$ (3)

Sometimes we use a single letter symbol, such as "p" or "q" to represent a whole phrase or sentence. Thus we may write:

"Two divides a and two divides b"

in the shorter form

"$2 \mid a$ and $2 \mid b$"

or replace this expression by the symbol "p" where

"p" means "$2 \mid a$ and $2 \mid b$".

Similarly we could use "q" to mean "$2 \mid a \cdot b$" or "two divides the product of a by b." Thus we can represent (3) by

If p, then q

We refer to "p" as the "hypothesis" and

We refer to "q" as the "conclusion."

In order to prove (3) we assume that p was true. That is, we assumed that the conjunction of "$2 \mid a$" and "$2 \mid b$" was true. Then, using our axioms and definitions, we proceeded to establish that the conclusion $2 \mid (a \cdot b)$ was true.

The direct method of proof is one of several accepted methods of establishing mathematical sentences as theorems. Often the direct method is not the simplest way to prove a sentence true. Another method of proof, called the indirect method, is useful in many instances. To illustrate the method we shall apply it to proving the following:

If $a \cdot b$ is an odd natural number, then a and b are both odd natural numbers. (4)

Proof:

As before, we begin by assuming that $a \cdot b$ is an odd natural number. [Note: we say that a natural number is odd if it is not even]. But rather than using this fact directly we now ask whether it is possible for one of a and b to be even? To answer this question we consider first the possibility that a is even. If a is even, $a = 2 \cdot x$, $x \in N$. Then, $a \cdot b = (2 \cdot x) \cdot b = 2 \cdot (x \cdot b)$ which means that $a \cdot b$ is even. But $a \cdot b$ is odd. Hence, a cannot be even, that is, a is odd. In a similar fashion we see that b cannot be even. Therefore, both a and and b must be odd if $a \cdot b$ is odd and our proof is complete.

In order to prove (4) we assumed that the hypothesis was true, that $a \cdot b$ was odd. Then we considered the possibility that the conclusion might be false, that is, that a was even or b was even. In either case this could not be true because it meant that $a \cdot b$ was even. We thus reasoned that the conclusion must be true.

Question: Can you justify each of the statements used in the proof of (4)?

The above proof concerning odd natural numbers made use of the definition of odd naturals as naturals which are not even. It is possible to give a more useful definition of odd natural numbers. For this definition we will need to review some ideas studied in your earlier work with arithmetic. In particular recall that when you were asked to divide a natural number by another natural number you frequently expressed the answer in terms of a *quotient* and a *remainder*. Consider the following two displays of work done to divide 15 by 2:

$$\begin{array}{r|l} & \;\;6 \\ \hline 2 & 15 \\ & \underline{12} \\ & \;\;3 \end{array} \qquad\qquad \begin{array}{r|l} & \;\;7 \\ \hline 2 & 15 \\ & \underline{14} \\ & \;\;1 \end{array}$$

In both displays we obtain a *quotient* and a *remainder*. On the left we have a quotient 6 and a remainder 3 whereas on the right we have a quotient 7 and a remainder 1. For the display on the left we have

$$15 = (6 \cdot 2) + 3$$

For the display on the right we have

$$15 = (7 \cdot 2) + 3$$

In a sense we have two "answers" for our division problem involving a quotient and a remainder. We resolve this situation of not having a unique solution by saying that we will accept that result in which the remainder is a whole number less than the divisor. Then the display on the left is unacceptable because the remainder 3 is not less than the divisor 2. Further, the display on the right is acceptable because the remainder 1 is a whole number than the divisor 2. The question of whether we can always find exactly one quotient and exactly one remainder when a whole number is divided by a natural number is answered by the following axiom which is known as the *Division Algorithm*.

A7. Let a be a whole number and b be a natural number.

Then there exist unique whole numbers q and r such that

$$a = (q \cdot b) + r \quad \text{with } 0 \leq r < b$$

Example 1: Let $a = 39$ and $b = 9$. Then the division algorithm (A7) guarantees that whole numbers q and r exist such that

$$39 = (q \cdot 9) + r \text{ with } 0 \leq r < 9$$

In fact if we let $q = 4$ and $r = 3$ we have

$$39 = (4 \cdot 9) + 3 \text{ with } 0 \leq 3 < 9$$

Moreover, the division algorithm guarantees that $q = 4$ and $r = 3$ are the unique whole numbers which satisfy

$$39 = (q \cdot 9) + r \text{ with } 0 \leq r < 9$$

Example 2: Consider a case where a is less than b.

If $a = 8$ and $b = 17$, then

$$8 = (0 \cdot 17) + 8$$

where the quotient is 0 and the remainder is 8. Note that the remainder is a whole number less than the divisor. That is $0 \leq 8 < 17$.

Example 3: If a whole number is divided by 2 the division algorithm guarantees that there exist unique whole numbers q and r such that

$$a = (q \cdot 2) + r \text{ where } 0 \leq r < 2$$

It is clear that the only possible values of r are 0 and 1. Thus we have

either $\quad a = (q \cdot 2) + 0 \quad$ (1)

or $\quad a = (q \cdot 2) + 1 \quad$ (2)

We can use the above to give us the following:

Definition 4: (a) n is an even whole number if and only if n can be expressed as $n = (q \cdot 2) + 0$ where q is some whole number.

(b) n is an odd whole number if and only if n can be expressed as $n = (q \cdot 2) + 1$ where q is some whole number.

It is easy to establish the following:

Let $E = \{x \mid x \text{ is an even natural number}\}$

and $\sigma = \{y \mid y \text{ is an odd natural number}\}$

Theorem (a) If $a \in E$ and $b \in \sigma$, then $(a + b) \in \sigma$

(b) If $a \in \sigma$ and $b \in \sigma$, then $(a + b) \in E$

(c) If $a \in E$ and $b \in \sigma$, then $(a \cdot b) \in E$

(d) If $a \in \sigma$ and $b \in \sigma$, then $(a \cdot b) \in \sigma$

The proof of the above will be called for in the exercises.

We conclude this discussion of odd and even natural numbers with a theorem whose proof makes use of Definition 4 and the above theorem. It also illustrates a method of proof sometimes called *proof by cases*.

Theorem: If n and $n + 1$ are natural numbers, then $n(n + 1)$ is an even natural number.

Proof: $n(n + 1) = n^2 + n$ (by A5 and by definition of n^2)

(1) If n is even, then n^2 is even. If n and n^2 are even, then $n^2 + n$, as the sum of two even natural numbers, is even.

(2) If n is odd, n^2 is odd, and if n and n^2 are odd, then $n^2 + n$, as the sum of two odd natural numbers, is even.

Hence, in either case (1) or (2) $n^2 + n$ is even. Since $n(n + 1) = n^2 + n$, $n(n + 1)$ is even.

Question: Why does the above proof consider only two cases?

11.4 Exercises

1. Complete the following:
   (a) $a = (q \cdot b) + r$, $0 \leq r < b$, is called the ____?____.
   (b) $(x + 1) \cdot y = x \cdot y + y$ follows from ____?____.
   (c) $7 \cdot 1 = 7$ follows from ____?____.
   (d) If $x = y$ and $p = q$, then $x + p = y + q$ follows from ____?____.
   (e) 7 is an odd natural number because ____?____.
   (f) If a is an odd natural number, then $a =$ ____?____.
   (g) If q is false implies p is false, then ____?____.
   (h) If $k \in N$ and $j \in N$, then $(k \cdot j) \in N$ follows from ____?____.
2. Find all possible pairs of whole numbers q and r such that $13 = (3 \cdot q) + r$. Which of these pairs are the quotient and remainder of the division algorithm? For which case(s) does r satisfy $0 \leq r < 3$?

3. (a) Prove if $3 \mid a$ and $3 \mid b$, then $3 \mid a + b$ where $a, b, \in N$

(b) Prove if $c \mid a$ and $c \mid b$, then $c \mid a + b$ where $a, b, c \in N$.

4. Prove if $a \mid b$ and $b \mid c$, then $a \mid c$ where $a, b, c \in N$.

5. Prove if $a \mid b$, then $a \mid bc$ where $a, b, c, \in N$.

6. Let E and σ represent respectively the set of even natural numbers and the set of odd natural numbers.

Prove (a) If $a \in E$ and $b \in \sigma$, then $(a + b) \in \sigma$

(b) If $a \in \sigma$ and $b \in \sigma$, then $(a + b) \in E$

(c) If $a \in E$ and $b \in \sigma$, then $(a \cdot b) \in E$

(d) If $a \in \sigma$ and $b \in \sigma$, then $(a \cdot b) \in \sigma$

7. If the natural number n is *not* a multiple of 3, then $n^2 + n$ *is* a multiple of 3. Prove the above theorem as follows: Assume $n^2 + n$ is *not* a multiple of 3 implies n *is* a multiple of 3.

8. Examine each of the statements (a), (b), and (c). If the statement is false then exhibit a counter example.
If the statement is true then list all the assumptions that you need in order to complete a proof of the statement.

(a) If $a \mid b$, then $a \mid b + c$

(b) If $a \mid b$, then $a \mid bc$

(c) If $a \mid b + c$ and $a \mid b$, then $a \mid c$.

9. In this problem we consider some tests that may be applied to divisibility questions involving base ten. These tests will generally fail when numbers are represented with numerals in bases different from ten.

Assume the following is true:

If $a \mid b_1, a \mid b_2, \ldots, a \mid b_{m-1}$ and
if $a \mid (b_1 + b_2 + \ldots b_{m-1} + b_m)$, then $a \mid b_m$.

Also note that any natural number N can be written in the form $N = a_n 10^n + a_{n-1} 10^{n-1} + \ldots + a_2 10^2 + a_1 10 + a_0$

(a) Prove that a natural number is divisible by 2 if and only if the last digit of its (base ten) numeral is even.

(b) Note $3 \mid (10-1)$, $3 \mid (10^2-1)$, $3 \mid (10^3-1)$, etc.. Assume $3 \mid (10^k-1)$ where k is any natural number. Prove a natural number is divisible by 3 if and only if the sum of the digits of its (base ten) numeral is divisible by 3. [Hint: $10^k = 10^k - 1 + 1$]

(c) Discover a decimal numeral test which indicates when a number is divisible by

(1) 4
(2) 5
(3) 6
(4) 8
(5) 9
(6) 10

(d) Prove any of the results you have discovered in (c).

11.5 Primes and Composites

It is obvious that the natural number 8 has more factors than the natural number 7. The set of factors of 8 is {1,2,4,8} whereas the set of factors of 7 is {1,7}. It is not hard to find other natural numbers like 7 which have exactly two distinct numbers in their factor set. For example, 11 is such a number since the set of factors of 11 is {1, 11}. 2 is another natural number with precisely two numbers in its set of factors. Such numbers as 2, 7, and 11 are called prime numbers. In general, we have the following:

Definition 4: A natural number is said to be a prime number if the number has two and only two *distinct* factors -- namely, 1 and the number itself.

Example 1: 3 is a prime number since the only factors of 3 are 1 and 3.

Example 2: 31 is a prime number since the only factors of 31 are 1 and 31.

Example 3: 91 is not a prime number because 91 = 7x13. That is, 91 has factors other than 1 and 91.

Example 4: 1 is *not* a prime number. What in the definition of prime number determines that 1 is not a prime?

We see from Example 4 that the least natural prime number is 2. What can we say about the primness or non-primeness of multiples of 2 which are greater than 2? We know that 4 is a multiple of 2. But 4 cannot be a prime number because it has a factor other than 1 and itself, namely 2. Similarly, 6, being a multiple of 2, has a factor 2 other than 1 and 6 and thus cannot be a prime number. In general, no multiple of 2 except 2 can be a prime number. Why?

What about multiples of the prime number 3? Can they ever be prime numbers? If we examine any multiple of 3 greater than 3, say 9 or 21 or 3000, we see that every such multiple has a factor other than 1 and itself, namely 3. In short, there are many natural numbers which are not prime. We call numbers of this type *composite numbers*. A composite number always has numbers in its factor set besides 1 and the number itself. The factor set for the composite number 9 is {1, 3, 9}.

Definition 5: A natural number is a *composite number*, if it is not equal to 1 and it is not a prime number.

Example 1: The natural number 51 is a composite number. Clearly 51 is not equal to 1. Also, 51 is not a prime number because it has the factors 3 and 17. We note that the factor set of 51, $\{1, 3, 17, 51\}$, has more than two elements.

Example 2: All multiples of 5, except 5, are composite. That is $\{10, 15, 20, 25, 30 \ldots\}$ consists of composite numbers. Why?

Example 3: The natural numbers 90, 91, 92, 93, 94, 95, 96, 98, and 99 are all composite. How would you check this? What can we say about 97?

From the remarks and examples above it can be seen that we now have a *partition* of the set of natural numbers into three disjoint subsets. These subsets are the following:

(i) the set consisting of 1 alone; that is $\{1\}$.

(ii) the set of prime natural numbers.

(iii) the set of composite natural numbers.

11.6 Exercises

1. Complete the following sentences:
   (a) If a natural number is a prime number, then its factors are ___?___.
   (b) If a natural number is not a prime number, then it is ___?___.
   (c) If a natural number is a prime number, then it has ___?___ elements in its set of factors.
   (d) If a natural number is not a prime number, then it has ___?___ elements in its factor set.
2. List the set of factors for the following natural numbers:
   (a) 10
   (b) 13
   (c) 12
   (d) 24
   (e) 34
   (f) 35
   (g) 36
   (h) 37
3. Determine which of the numbers given in Exercise 2 are
   (a) prime
   (b) composite
   (c) both prime and composite
4. What can be said about every multiple of a prime number which is greater than that prime number?
5. (a) What is the greatest prime number less than 50?
   (b) What is the least composite number?
6. What can be said about the product of two prime numbers?
7. (a) List the set of all even prime numbers.
   (b) List the set of all odd prime numbers less than 20.
8. Re-examine the definition of composite number. Can you formulate a different definition which makes use of the term "factor" or "factor set"?

11.7 Complete Factorization

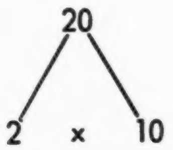
As you continue your study of the set of natural numbers and their properties you will frequently have to examine the factors that make up the product expressions of a natural number. What can we say about the factors that make up the product expressions of prime numbers? We have seen that

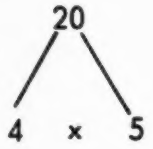
$$2 = 1 \cdot 2$$
$$3 = 1 \cdot 3$$
$$5 = 1 \cdot 5, \text{ etc.}$$

By the definition of prime numbers the only factors a prime p has are 1 and p. However, we find that every composite number can be renamed as a product expression other than 1 times the number. For example, 20 can be renamed using either of the following product expressions:

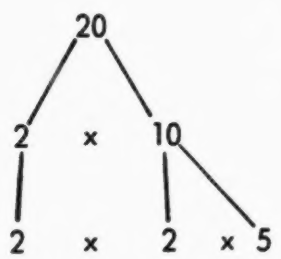
$$2 \cdot 10 \quad (1)$$
$$4 \cdot 5 \quad (2)$$

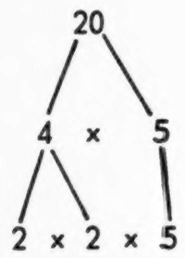
These product expressions of 20 can be shown in another way:





On the left we have a tree diagram to represent (1) and on the right a tree diagram to represent (2). It is possible to continue each of the above diagrams by completing another row to indicate product expressions of 20 as follows:



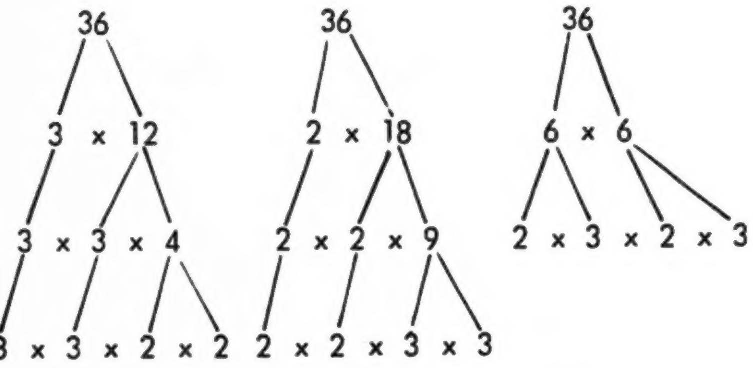


We see that every number named in the last row of both diagrams is a prime number. (We shall refer to such tree diagrams as *factor trees*.) Moreover, the last rows in oth factor trees contain exactly the same prime num-ers. Thus, starting with either of the product expres-ions (1) and (2) of 20 we obtain exactly the same pro-uct expression of 20. In this case we see that 20 has product expression such that each factor that makes p the product expression is a prime number. We shall escribe this situation by saying that 20 can be xpressed as a *product of prime factors*.

Our attention is directed to the following questions:

Can every composite number be expressed as a pro-uct of prime factors? In other words, does there exist product expression for each composite number in hich each factor is a prime number? Furthermore, is nere only one such product expression?

The following factor trees for 36 suggest that the nswer to the above questions should be "Yes."



'e note again that the last row in each of the above actor trees is a product expression for 36 in which ach factor is a prime number. Moreover, the same set f factors appear in each product expression. Note hat the order of the factors in each of the last rows f the factor trees is different. Is this change in the he order of the factors a signifigant change? The nswer is "No." Because of the commutative property f multiplication in (N, x), the fact that they are rranged in different order is immaterial. Thus, using xponents, we can express the last row in each of the bove tree diagrams as

$$2^2 . 3^2$$

When a composite number is expressed as a product f prime factors, we refer to this as a *complete factor-zation* of the given number.

The following are examples of complete factorizat-ons:

$$72 = 2 \cdot 36$$
$$= 2 \cdot 2 \cdot 18$$
$$= 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3$$

$$182 = 2 \cdot 91$$
$$= 2 \cdot 7 \cdot 13$$

$$150 = 2 \cdot 75$$
$$= 2 \cdot 3 \cdot 25$$
$$= 2 \cdot 3 \cdot 5 \cdot 5$$

Notice that when each factor in the final product expression is a prime number then we say that the product expression for complete factorization has been found.

One important question that can be asked is the following: If a composite number has a complete factorization, could it have a second complete factorization involving different prime numbers? All the examples considered above seem to indicate that there is only one complete factorization for a given composite number. For example consider

$$150 = 2 \cdot 3 \cdot 5 \cdot 5$$

If you experiment with other possible prime factors, such as 7, 11, 13, etc., you will find that the above is the only complete factorization of 150.

The above examples illustrate one of the most important and fundamental properties of the set of natural numbers. The property is called *The Unique Factorization of the Natural Numbers:*

> Every natural number greater than 1 is either a prime or can be expressed as a product of primes in one and only one way except tor the order in which the factors occur in the product.

We shall see how this property can be used to solve, in a new way, a problem that you met earlier in this course.

There was an exercise in Chapter 2 [See Section 2.2, Exercise 9, p. 32] in which you were to find the *greatest common divisor* of 24 and 16. It turns out that finding the greatest common divisor of two natural numbers is equivalent to finding the greatest common factor of the two numbers. We can redefine a greatest common divisor of two natural numbers using the terminology of this chapter.

Definition 6: The *greatest common divisor* (abbreviated g.c.d.) iated g.c.d.) of two natural numbers, a and b, is the largest natural number d such that d | a and d | b. d is written as g.c.d. (a, b) or d = (a, b).

In Chapter 2 you found g.c.d. (24, 16) essentially as follows:

Consider the set made up of the factors of 24, which we will call A: A = {1, 2, 3, 4, 6, 8, 12, 24}

The set of factors of 16 we will call B:

Then A ∩ B = {1, 2, 4, 8} is the set of common factors (divisors) of 16 and 24. Clearly 8 is the greatest common divisor of 24 and 16. That is g.c.d. (24, 16) = 8. We see that 8 is the greatest natural number such that 8 | 24 and 8 | 16.

Question: Why will 1 always be an element in the intersection of the factor sets of two natural numbers?

A second solution to the above problem is as follows: By the Unique Factorization Property we know that both 24 and 16 can be expressed as a product of primes where the factors of the product are unique. In fact we have $24 = 2 \cdot 2 \cdot 2 \cdot 3$ and $16 = 2 \cdot 2 \cdot 2 \cdot 2$. We see that the product expression $2 \cdot 2 \cdot 2$ is common to both factorizations and yields the greatest common divisor 8. This technique is useful when the numbers are small. For example to find g.c.d. (45, 108) we determine that

$$45 = 3^2 \cdot 5$$

$$\text{and } 108 = 2^2 \cdot 3^3$$

We see that 3 is a common factor. However, $3^2 = 9$ is also a common factor and is the greatest common factor of 45 and 108.

11.8 Exercises

1. Factor the numbers listed in as many ways as possible using only two factors each time. We shall say that $2 \cdot 3$ is not different from $3 \cdot 2$ because of the commutative property of multiplication in $(N, \cdot)$.

(a) 9
(b) 10
(c) 15
(d) 100
(e) 24
(f) 16
(g) 72
(h) 81

2. Write a complete factorization of:

(a) 9
(b) 10
(c) 15
(d) 100
(e) 24
(f) 16
(g) 81
(h) 210
(i) 200
(j) 500

3. What factors of 72 doe not appear in a complete factorization of 72?

4. What will be true about the complete factorization of every

(a) even natural number

(b) odd natural number

5. Construct at least two tree diagrams for each of the following:

(a) 24
(b) 96
(c) 625
(d) 1000

6. Find the greatest common divisor of the following pairs of numbers by making use of their complete factorizations.

(a) 70 and 90
(b) 80 and 63
(c) 372 and 390
(d) 663 and 1105

7. Determine if g.c.d. is a binary operation on N. If it is, explore its properties. If it fails to be a binary operation on N, explain why it does fail.

8. Copy the following tables for natural numbers and complete it through $n = 30$.

n	Factors of n	Number of factors	Sum of factors
1	1	1	1
2	1,2	2	3
3	1,3	2	4
4	1,2,4	3	7
5	1,5	2	6
6	1,2,3,6	4	12
7	1,7	2	8
8	1,2,4,8	4	15

(a) Which numbers represented by n in the table above have exactly two factors?

(b) Which numbers n have exactly three factors?

(c) If $n = p^2$ (where p is a prime number), how many factors does n have?

(d) If $n = pq$ (where p and q are prime numbers and not the same), how many factors does n have? What is the sum of its factors?

(e) If $n = 2$ (where k is a natural number), how many factors does n have?

(f) If $n = 3$ (where k is a natural number), how many factors does n have?

(g) If $n = p$ (where k is a natural number and p is a prime), how many factors does n have?

(h) Which numbers have 2n for the sum of their factors? (These numbers are called *perfect numbers.*)

9. If we list the set of multiples of 30, we obtain {30, 60, 90, 120, 150, 180, ...}. Also, if we list the set of multiples of 45, we obtain {45, 90, 135, 180, 225, 270, ...}. We see that a common multiple of 30 and 45 is 180. However, there is a common multiple which is the least common multiple of 30 and 45; namely 90. We write this as l.c.m. (30,45) = 90.

(a) Examine the complete factorizations of 30 and 45 and explain how one could use these to find that the least common multiple of 30 and 45 is 90.

(b) Similarly, find the least common multiples of the following numbers by making use of their complete factorizations:

(1) 30 and 108
(2) 45 and 108
(3) 15 and 36
(4) 81 and 210
(5) 16 and 24
(6) 200 and 500

(c) Can you find any relationship between the greatest common factor (g.c.f) of a and b and the least common multiple (l.c.m.) of the same a and b? Experiment and write a report on your findings.

10. Determine if l.c.m. is a binary operation on N. Write a report of your findings.

11.9 The Sieve of Eratosthenes

The fact that every composite number can be expressed as a product of primes in one and only one way, except for order, indicates that the set of prime numbers are the basic elements, the atoms so to speak, in the structuring of the natural numbers by multiplication. If we wish to have a basic understanding of multiplication of natural numbers (and division, which is defined in terms of multiplication), then it is to our advantage to be aware of some properties of the set of prime numbers.

A list of all the primes up to a given natural number N may be constructed as follows: Write down in order all the natural numbers less than N. In Figure 11.1 we have done this for N = 52. Then strike out 1 because by definition it is not a prime. Next, encircle 2 because it is a prime number. Then strike out all remaining multiples of 2 in the list, that is, 4, 6, 8, 10, etc. such multiples of 2 are, as we discussed earlier, composite numbers.

Next encircle 3, the next number we encounter in our list. After 3 is encircled, we strike out 6, 9, 12, ..., that is, all multiples of 3 remaining in the list. (Note that 6 was struck out when we considered multiples of 2 and also when we considered multiples of 3.) In a similar way we continue this process by next encircling 5 and striking out its remaining multiples. Lastly we encircle 7 and strike out its remaining multiples.

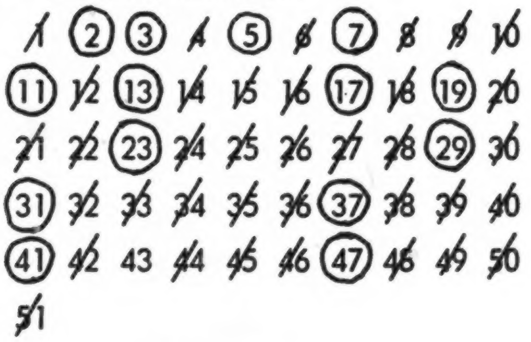


Figure 11.1

Note that if we encircle all the numbers remaining in the list we obtain all the natural prime numbers less than N = 52. In all there are 15 such prime numbers obtained by this process, known as the Sieve of Eratosthenes. The sieve catches all the primes up to N in its meshes.

Complete tables of all primes less than 10,000,000 have been computed by this method and refinements of this method. Such tables are useful in supplying data concerning the distribution and properties of the primes.

Even the small list constructed above gives some indication that the primes are not distributed in any sort of obvious way among the natural numbers. Also, we see that it may happen that a number, p, is a prime and p + 2 is also a prime. Such pairs of primes are called *twin primes*. Examples of twin primes in the list above include 11 and 13, 17 and 19, 29 and 31, 41 and 43.

11.10 Exercises

1. (a) In the above list, what was the first number struck out when we sieved for the following:

   (1) multiples of 2 (3) multiples of 5

   (2) multiples of 3 (4) multiples of 7

   (b) Can you make a conjecture concerning the first number struck out if we sieve for multiples of a prime p?

   (c) Explain why we did not have to sieve for multiples of the prime 11?

   (d) What is true of all numbers that

   (1) pass through the sieve?

   (2) remain in the sieve?

   (e) Would any new numbers be crossed out if we sieved for multiples of 4? Why or why not?

2. Make up a list of natural numbers less than 131.

   (a) Carry out the Sieve of Eratosthenes process on this set of numbers.
   (b) How many primes are there less than 101?
   (c) How many primes are there less than 131?

   (d) What is the largest prime number in your list?

   (e) What is the largest prime, p, for which you had to determine multiples in the sieving process? Explain.

3. (a) List the pairs of prime numbers less than 100 which have a difference of 2.

   (b) What name is given to such pairs?

   (c) How many such pairs are there less than 100?

4. Make up a list of numbers which goes from 280 through 290.

   (a) Apply the Sieve of Eratosthenes process to this list.

   (b) List all the primes obtained from this sieving.

   (c) For which primes did you have to seek multiples?

   (d) Explain why you selected a certain prime as the largest for which you sought multiples.

5. (a) List the triplets of prime numbers less than 131 which have a difference of 2. Such triplets are called *prime triplets*.

   (b) After you have found the smallest set of prime triplets, explain why no other distinct set of prime triplets could have 3 as a factor.

   (c) Assume that there is a second set of prime triplets. Call them p, p + 2, p + 4. From (b) we know that $p \neq 3k$ where k is some natural number. Why?

(d) If $p \neq 3k$, then what is the remainder obtained when p is divided by 3?

(e) Can you examine $p + 2$ and $p + 4$ and prove that p, $p + 2$, and $p + 4$ do not exist as primes?

(f) What conclusion can you draw from (a) – (e)?

11.11 On the Number of Primes

Euclid (circa 300 B. C.) answered the following question: Is there a finite or a non-finite number of prime numbers? As you work with the sieve of Eratosthenes you probably note that as you continue sieving the primes become relatively scarce. However, Euclid proved that, as one continues to examine the set of natural numbers, primes will always be encountered if we seek long enough. He proved that there are a non-finite number of primes.

Euclid's argument proceeds as follows: Assume there is a largest prime. Let us denote this largest prime as "P". All the primes can then be written in a finite sequence

$$2, 3, 5, 7, \ldots, P.$$

Since P is the largest prime, all numbers greater than P must be composite; that is, every number greater than P must be divisible by at least one of the primes in the above sequence (Why?). But now consider the number

$$N = (2\cdot3\cdot5\cdot7\cdot \ldots \cdot P) + 1.$$

that is, the number obtained by adding 1 to the product of all the primes. Since N is greater than P, it must be a composite number, and therefore divisible by at least one of the primes in the above sequence. But by which? It can be argued that N is not divisible by any of the primes 2, 3, 5, 7, ..., P (Why?). Hence N cannot have any prime factors, which contradicts the fact that N is composite. Therefore, the assumption that the number of primes is finite leads to a contradiction, and we must conclude that there are a non-finite number of primes.

It is interesting to note that the number of prime twins is not known! Unlike the situation for the primes, efforts to determine the number of such prime twins have not proved successful.

Another famous unsolved problem also deals with primes. It is called Goldbach's Conjecture. Goldbach stated, in a letter to Euler in 1742, that in every case that he tried he found that any even number greater than 2 could be represented as the sum of two primes. For example, $4 = 2 + 2$, $6 = 3 + 3$, $8 = 5 + 3$, etc. No one has ever been able to prove or disprove this conjecture of Goldbach. The problem posed in the conjecture is interesting because (1) it is easily stated and (2) it involves addition whereas primes are defined in terms of multiplication. In any case, it has resisted solution for over two hundred and twenty years.

11.12 Exercises

1. Show that the following numbers all satisfy Goldbach's Conjecture.

(a) 10
(b) 12
(c) 14
(d) 16
(e) 18
(f) 20
(g) 36
(h) 48
(i) 100
(j) 240

2. In working with Euclid's proof that the set of primes is non-finite we find that possible values of N include: $2 + 1$, $2\cdot3 + 1$, $2\cdot3\cdot5 + 1$, $2\cdot3\cdot5\cdot7 + 1$, $2\cdot3\cdot5\cdot7\cdot11 + 1$, $2\cdot3\cdot5\cdot7\cdot11\cdot13 + 1$, $2\cdot3\cdot5\cdot7\cdot7\cdot11\cdot13\cdot17 + 1$, etc.

(a) Explain how each of the numbers in the above list was formed. In each case what is P? What is N?

(b) The first 5 numbers in the list are primes. Compute them and verify that at least 4 of them are in fact primes.

(c) Note that $2\cdot3\cdot5\cdot7\cdot11\cdot13 + 1 = 30031$ and this number is composite because $30031 = (59)(509)$. Verify this.

(d) Prove that $2\cdot3\cdot5\cdot7\cdot11\cdot13\cdot17 + 1$ is a composite number. (Hint: be efficient!)

(e) Discuss Euclid's argument with regard to the number shown in (d).

(f) Answer the two questions. "Why?" given in Euclid's proof of the infinitude of the primes.

(g) Explain why a computer could never settle the question concerning the number of prime twins.

11.13 Euclid's Algorithm

We have seen that one way to find the g.c.d. of two natural numbers is to begin by expressing each of the numbers as a product of prime factors. However, this is not practical when the numbers considered are quite large. A method which is often used to find the g.c.d. of large numbers is based on repeated use of the division algorithm.

We illustrate this by considering the problem of finding the g.c.d. of 28 and 16. By applying the division algorithm we have

$$28 = (1\cdot16) + 12 \text{ where } 0 \leq 12 < 16$$

Note that if $a|b + c$ and $a|b$, then $a|c$. Thus any number that divides 28 and 16 must also divide 12. Thus the g.c.d. (28,16) must divide 12. Let g.c.d. $(28,16) = d$. Then $d|12$ implies d is a common divisor of 16 and 12.

Also

$d = \text{g.c.d.}\,(16,12)$

because if there was a larger divisor of 16 and 12 it would divide 28 and then d would not be the g.c.d. (28,16). Hence, we have g.c.d. (28,16) = g.c.d. (16,12). We continue the process by using the division algorithm again to obtain

$16 = (1\cdot 12 + 4$ where $0<4<12$

By the same argument as above we have g.c.d. (16,12) = g.c.d. (12,4) Therefore, g.c.d. (28,16) = g.c.d. (12,4). Lastly, we apply the division algorithm to obtain

$12 = (3\cdot 4) + 0$

and we see that the g.c.d. (12,4) = 4
Thus g.c.d. (28,16) = 4.
The following example illustrates the algorithm indicated above:

Example: Find the g.c.d. of 7469 and 2387

$7469 = (2387)(3) + 308$ g.c.d. (7469,2387) = g.c.d. (2387,308)

$2387 = (308)(7) + 231$ g.c.d. (2387,308) = g.c.d. (308,231)

$308 = (231)(1) + 77$ g.c.d. (308,231) = g.c.d. (231,77)

$231 = (77)(3)$ g.c.d. (231,77) = 77

Thus g.c.d. (7469,2387) = 77.

Note that we first divide the larger number, 7469, by the smaller number, 2387, and find the *remained*, 308 (which is less than the smaller number). Next we divide the smaller number by this remainder 308 and find a new remainder 231. Now we divide the first remainder 308 by the new remainder 231 and find the third remainder, 77. We continue this division until we obtain a remainder 0. The last non-zero remainder thus found is the g.c.d.

The procedure used to obtain the set of equations that is obtained by successive applications of the division algorithm is known as *Euclid's Algorithm.*

It can happen that when we find the g.c.d. of two numbers it turns out to be 1. For example, it is clear that

g.c.d. (5,13) = 1

and with a little work we can see that

g.c.d. (124,23) = 1

Such pairs of numbers whose g.c.d. is 1 play an important role in Number Theory.

Definition 7: If the greatest common divisor of two natural numbers *a* and *b* is 1, we say that *a* and *b* are relatively prime.

Thus 5 and 13 are relatively prime since g.c.d. (5, 13) = 1. Similarly 124 and 23 are relatively prime. We shall use the idea of two numbers being relatively prime in our next axiom.

A8. If d = g.c.d. (a,b), then there exist integers x and y such that

$$d = x \cdot a + y \cdot b$$

In particular, if a and b are relatively prime, there exist integers x and y such that $x \cdot a + y \cdot b = 1$.

Example 1: g.c.d. (72,86) = 2 and $2 = 6(72) + (-5)(86)$

Here x = 6 and y = -5

Example 2: g.c.d. (5, 7) = 1

and $1 = 3\,(7) + (-4)\,(5)$

Here x = 3 and y = -4.

Example 3: g.c.d. (147,30) = 1

and $1 = 23\,(147) + (-26)\,(130)$

Here x = 23 and y = -26.

In order to prove an important theorem we need only the underlined portion of A8 (which is illustrated in Examples 2 and 3 above). The following theorem will allow us to prove a number of theorems that tie together the ideas of "prime" and "divisibility."

Theorem: If $a \mid bc$ and g.c.d. (a,b) = 1, then $a \mid c$.

Proof: Since g.c.d. (a,b) = 1, then, by A8

$$1 = ax + by$$

where x and y are integers. Then c = c we have, by Theorem A, $c \cdot 1 = c\,(ax + by)$. Applying A6 on the left and A5 on the right, we have

$$c = cax + cby$$

By hypothesis $a \mid bc$ which by A3 implies $a \mid c \cdot b$. But $a \mid cb$ implies $a \mid cby$. (Why?) Similarly $a \mid cax$. Thus, we conclude that $a \mid c$. (Why?)

Example 1: $7 \mid 70$. Consider 70 as 5 (14). Then we have $7 \mid 5\,(14)$ and g.c.d. (7,5) = 1. Hence by the above theorem $7 \mid 14$.

Example 2: $10 \mid 840$. Consider 840 as 21 (40). Then we have $10 \mid (21)\,(40)$ and g.c.d. (10,21) = 1. Hence $10 \mid 40$.

Among the theorems that are easily established using the above theorem are:

(1) Let p be a prime such that $p \mid bc$ and $p \mid b$. Then $p \mid c$.
(2) If p is a prime and $p \mid ab$, then either $p \mid a$ or $p \mid b$ (or both).

11.14 Exercises

1. Using the Euclidean Algorithm find the greatest common divisor of each of the following pairs of numbers.

(a) 1122 and 105

(b) 2244 and 418

(c) 220 and 315

(d) 912 and 19,656

2. Find the g.c.d. (144, 104) using two different methods.
3. (a) What is the g.c.d. of a and b if a and b are distinct primes?
   (b) If a is a prime and b is a natural number such that $a \mid b$ what is the g.c.d. (a,b)?
4. Prove the following:

   Let p be a prime such that $p \mid bc$ and $p \mid b$. Then $p \mid c$.
5. Prove: If p is a prime and $p \mid ab$ then either $p \mid a$ or $p \mid b$ (or both)
6. Prove: If a and b are relatively prime and $a \mid c$ and $b \mid c$, then $ab \mid c$.
7. Prove: If d = g.c.d. (a,b) and a = rd and b = sd, then r and s are relatively prime.
8. Construct a flow chart for finding the g.c.d. of a and b by the Euclidean Algorithm.
9. *Fermat's Little Theorem.* In the year 1640 Fermat stated the following: If p is a prime that is not a divisor of the natural number a, then $p \mid (a^{p-1} - 1)$.
   (a) Find two examples which illustrate this theorem.
   (b) Note that there is the restriction that $p \nmid a$. What would follow if $p \mid a$?
   (c) What can we conclude if p is not a prime?
   (d) Can you prove Fermat's Little Theorem?

11.15. Well-Ordering and Induction.

We have stated thus far eight axioms, A1-A8, which are basic properties of (N, +, ·). These have enabled us to investigate many interesting problems in number theory and to prove several theorems about the natural numbers. Perhaps you have noticed that several of the basic properties of the natural numbers are shared by other operational systems. For instance, A1-A6 are also properties of $(Z_5, +, \cdot)$. But (N, +, ·) is quite different in other respects from $(Z_5, +, \cdot)$. Let us look at some further properties of (N, +, ·) that distinguish it from other operational systems.

The first of these properties is the *Well-Ordering Axiom.* You will recall that if a and b are any two natural numbers we say that a is smaller than b, or that $a<b$, if and only if there is a natural number c such that $a + c = b$. Thus, given any two different natural numbers a smallest member of the pair may be determined. It is easy to see that this is also true for any set of three natural numbers. But is this true for *any* non-empty set of natural numbers? That is, does any non-empty set of natural numbers contain a smallest member? Let us consider the following examples:

Example 1. The set of even natural numbers. The smallest member of this set is 2.

Example 2. The set of odd natural numbers. The smallest member of this set is 1.

Example 3. The set of natural numbers which divide 1001. The smallest member of this set is 7. You should check this result for yourself.

It seems reasonable that the answer to our question in general should be "yes", and this is the content of the *Well-Ordering Axiom,* A9.

A9. Every non-empty set of natural numbers contains a smallest natural number.

To see that this is a distinctive property of the natural numbers not enjoyed by other sets of numbers we need only examine subsets of the integers Z. Not every non-empty subset of Z contains a smallest integer. For instance, the set of negative integers contains no smallest integer, since $-1 > -2 > -3 > -4 > -5 >$ Another example is the set of integers which have remainder 3 when divided by 5. This set is $\{\ldots, -17, -1$ $-7, -2, 0, 3, 8, 13, 18, \ldots\}$. We will see later that the axiom also does not hold for the set of rational number

The second property of the natural numbers that we consider is the Induction Axiom. If we begin with 1, and continue adding 1, we obtain the sequence $2 = 1 + 1$, $3 = 1 + 1 + 1$, $4 = 1 + 1 + 1$, ... In this we can eventually attain any natural number n from the natural number 1.

Looking at the situation another way, we let D be subset of N and ask whether or not D is a proper sub of N. If D is not a proper subset of N we know that D = N. Now suppose $1 \in D$ and, in general, whenever natural number $k \in D$, the natural number $(k + 1) \in D$ also. Then, since $1 \in D$, $1 + 1 = 2 \in D$, $2 + 1 = 3 \in D$, etc. It seems reasonable to assume that under these conditions every natural number is in D. That is, D This is our next axiom, the Induction Axiom, A10.

A10. Given a set D of natural numbers such that

1) $1 \in D$

2) $k \in D \rightarrow (k + 1) \in D$,

we many conclude that D = N.

This axiom is the basis for a powerful method of pro ing sentences about natural numbers. Let us recall a few ideas concerning open sentences and statements An open sentence in one variable cannot be asserted be either true or false. However, if we make some assertion as to what the variable represents, they be come statements that are either true or false. For example, "$x + 4 = 2x + 2$" is an open sentence and he s neither true nor false. However, "for every natural umber x, $x + 4 = 2x + 2$" is a false statement. (Try $= 1$). Note that the statement "there exists at least ne natural number such that $x + 4 = 2x + 2$" is a true tatement. Why? Other examples of statements on N are:

Example 1: For every natural number n greater than 2, 2 is a factor of $n^2 + n$.

Example 2: For every natural number n, $n^2 - n + 41$ is a prime number.

Example 3: For every natural number, the sum of the first n odd positive integers is equal to n^2

Example 4: There is no natural number n such that the sum of its factors is $2n + 1$.

Example 5: There exists a natural number n such that $n > 3$ and $n < 9$.

The solution set of the sentence in example 5 is clearly

$$\{4, 5, 6, 7, 8\}$$

We also call this set the *truth set* of the sentence in Example 5. It is clear that the statement in Example 5 is true.

In general, the truth set of a sentence in one variable is the set of all numbers and only those numbers, in the domain of the variable which make the sentence true. We often use the notation "P(n)" (read "P of n") to represent an open sentence. If P (n) denotes the open sentence "$n^2 - n + 41$ is a prime number" in Example 2, then we see that P(1) is true since "$1^2 - 1 + 41$ is a prime number." However P(41) is false. Show this! We conclude that the statement in Example 2 is false. If one experiments with the problem posed in Example 4 he soon finds that it is quite a hard problem. In fact, it is a good example of an easily stated but unsolved problem in number theory. One attempt to settle the problem would be to find a counter-example. We see that $n = 8$ fails as a counter-example because the sum of the factors of 8 is 15, whereas letting $n = 8$ we have $2(8) + 1 = 17$.

It is possible to get quite close to our goal. For example, let $n = 28$. The numbers which divide 28 are 1, 2, 4, 7, 14, and 28. The sum of these numbers is 56. But 56 is not equal to $2n + 1 = 2(28) + 1 = 57$. In the two cases attempted, we have failed to find a value of n which contradicts the condition of the problem.

But everyone who has ever tried to solve the problem has failed. Thus we do not know if the statement given in Example 4 is true or false.

Let us examine the open sentence in Example 3 more closely. We are to consider the sum of the first n odd natural numbers. We have

$$P(1) = 1 \qquad = 1$$
$$P(2) = 1 + 3 \qquad = 4$$
$$P(3) = 1 + 3 + 5 \qquad = 9$$
$$P(4) = 1 + 3 + 5 + 7 \qquad = 16$$
$$P(5) = 1 + 3 + 5 + 7 + 9 = 25$$

Notice that the sum of the first 5 odd natural numbers is $5^2 = 25$. The above results certainly suggest that the statement in Example 2 is well founded. It appears that if we let N be any natural number that

$$P(N) = 1 + 3 + 5 + 7 + \dots + (2N-1) = N^2$$

To prove the statement of Example 2, as it is rephrased in the previous paragraph, is true for every $n \in N$, we must show that its truth set, T, is the set of natural numbers N, or that $T = N$. It is in this situation that AIO is useful as a tool of proof. We shall make no attempt to carry out such a proof but simply indicate how AIO applies.

First, we must show that $1 \in T$, or that P(1) is true. This has already been done in previous discussion. Second, we must show that if $k \in T$ then $k + 1 \in T$. That is, if P(k) is true then P(k + 1) is true. Then the Induction Axiom allows us to conclude that $T = N$ or that P(n) is true for all $n \in N$.

11.16 Exercises

1. What does the Well-Ordering axiom assert about each of the following:
   (a) $\{4, 5, 6, 7, 8\}$
   (b) the set of prime natural numbers
   (c) the empty set
2. Can you make a conjecture concerning the sum of the first n natural numbers?
   Consider $1;\ 1 + 2;\ 1 + 2 + 3; \dots ;\ 1 + 2 + \dots + n$.
3. (a) Can you make a conjecture concerning
      (1) the sum of the first n even natural numbers
      (2) the sum of the first n^3 natural numbers (that is $1^3 + 2^3 + \dots + n^3$)
      (3) the sum of the first n^2 natural numbers
   (b) Can you find a relationship between the sum of the first n natural numbers and the sum of the first n^3 natural numbers?
4. Consider $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \dots + n(n + 1)$
   Can you make a conjecture concerning this sum?

11.17 Summary

In this chapter we have explored topics in number theory. You have had an opportunity to make conjectures and then to prove your conjectures.

At this time you should be able to give a clear description of what is meant by *factor, multiple, prime number, composite number, even and odd natural numbers, greatest common divisor*, and *complete factorization*. Can you state the Unique Factorization Property of the natural numbers?

You saw that the Sieve of Eratosthenes provides one way to determine primes up to some finite number. Do you believe that this is an efficient tool for finding primes? Can you describe several ways of finding the g.c.d. of two natural numbers? What purpose did Euclid's Algorithm serve and on what principle was it based? What is meant by the Well-Ordering and Induction Axiom? Can you state some properties of prime numbers? Can you state some problems that no one has ever been able to solve?

Overall, your awareness of the set of natural numbers should be increased. Also you should be more aware of what constitutes a proof in mathematics and the fact that there are varying methods of proving theorems.

11.18 Review Questions

1. Explain why the following are true.
   - (a) 10 is a factor of 50
   - (b) 30 is a multiple of 6.
   - (c) 6 is a factor of 30
   - (d) 6 is a factor of 6
   - (e) 7 is not a factor of 30
   - (f) 7 is a prime number
   - (g) 6 is a composite number
   - (h) 91 is a composite number
2. Define the following terms
   - (a) factor
   - (b) multiple
   - (c) prime
   - (d) composite
3. Give a complete factorization of each of the following:
   - (a) 38
   - (b) 72
   - (c) 96
   - (d) 97
4. Using the data in 3 above, determine
   - (a) g.c.d. (38, 72)
   - (b) g.c.d. (38, 96)
   - (c) g.c.d. (72, 96)
   - (d) g.c.d. (72, 97)
5. Using the data obtained in 3, determine
   - (a) l.c.m. (38, 72)
   - (b) l.c.m. (38, 96)
   - (c) l.c.m. (72, 96)
   - (d) l.c.m. (72, 97
6. Using the Sieve of Eratosthenes process determine all primes between 130 and 150.
   - (a) How many primes are in this set of numbers?
   - (b) How many twin primes are in this set?
   - (c) What is the largest prime p for which you have to determine multiples to find all the primes in this set of numbers?
7. Using the Euclidean Algorithm check one of your answers for 4 (c) above.
8. *Prove:* if $a \mid b$ and $b \mid c$, then $a \mid c$ where $a, b, c \in N$.
9. If $9 \mid n$ and $10 \mid n$ does it follow that $90 \mid n$? Explain.
10. Prove if $a \mid b$ where a is a prime, then g.c.d. (a, b) $= 1$.
11. Discuss what insights into $(N, +, \cdot)$ are provided, for you, by the Well Ordering Axiom and the Induction Axiom.

CHAPTER 12 THE RATIONAL NUMBERS

12.1 Operations on Z: Looking Ahead.

What is the solution of the equation

$$5 + x = -3?$$

As we learned in Chapter 4, the solution is

$$-3 - 5,$$

or -8. The number -8 is called the *difference* between -3 and 5, or the result of *subtracting* 5 *from* -3. Suppose that two integers, a and b, are selected, and the following equation written:

$$b + x = a.$$

Do we know that this equation has a solution? From our previous work, we know that the solution is

$$a - b,$$

regardless of what the integers a and b are. Thus, the solution of "$b + x = a$" is the difference between a and b, or the result of subtracting b from a. Since this is true for any pair of integers, we know that given any two integers, there is another integer which is their difference. The following table, which shows a number of particular cases, should make this clear.

Equation	Solution	Ordered Pair	Subtraction Assignment
$5 + x = -3$	-8	$(-3,5)$	$(-3,5) \longrightarrow -8$, or $-3 - 5 = -8$
$3 + x = 7$	4	$(7,3)$	$(7,3) \longrightarrow 4$, or $7 - 3 = 4$
$8 + x = 2$	-6	$(2,8)$	$(2,8) \longrightarrow -6$ or $2 - 8 = -6$
$-4 + x = 9$	13	$(9,-4)$	$(9,-4) \longrightarrow 13$, or $9 - (-4) = 13$
$b + x = a$	$a - b$	(a,b)	$(a,b) \longrightarrow a - b$

From the above discussion, do you see that *subtraction is a binary operation* on the set Z of integers? (If you have forgotten the definition of a binary operation on a set, see Section 2.3.)

Now consider an equation of the type

$$b \cdot x = a,$$

where a and b are integers. Do we know that this equation has a solution in Z, regardless of what the integers a and b are? To answer this question, study the following two examples.

Example 1. Let $a = -12$, and $b = 3$. Then the equation is $3 \cdot x = -12$. Since we know that $3 \cdot (-4) = -12$, we certainly have an integer, -4, as a solution. And because this is true, we say that -4 is the *quotient* of -12 and 3, or the result of *dividing* -12 by 3.

Thus, $-12 \div 3 = -4$, or $\dfrac{-12}{3} = -4$.

Using Example 1 as a guide, if there is an integer x such that

$$b \cdot x = a,$$

then we say that x is the quotient of a and b, or the result of dividing a by b. Furthermore, we write either of the following:

$$x = a \div b: \quad x = \frac{a}{b}.$$

Example 2. Let $a = -10$, and $b = 3$. Then the equation is $3 \cdot x = -10$.

Do you see that there *is no integer* which is a solution of this equation? That is, there is no integer which is the quotient of -10 and 3, and the symbol "$-10 \div 3$" does not name an integer. Also, division is *not* a binary operation on Z. (Why not?)

We now know (from Example 2) that there are equations of type

$$b \cdot x = a$$

where a and b are integers, that do not have an integer for a solution. We have been in this kind of predicament earlier. For instance, the equation

$$3 + x = 2$$

has no *whole number* for a solution. But with the introduction of some new numbers, the integers, there is a solution, namely -1. One of our purposes in this chapter is to try to introduce still another set of numbers so that an equation such as "$3 \cdot x = -10$" will have a solution.

12.2 Exercises.

1. For each of the following equations, give the solution (in the set Z of integers), fill in the difference of the two numbers, and then show the assignment which subtraction makes to the given ordered pair of integers. The first row has been completed correctly.

Equation	Solution	Subtraction Assignment
$3 + x = -2$	-5	$-2 - 3 = -5$ or $(-2,3) \longrightarrow -5$

$x + 4 = 6$

$x + 4 = 1$

$312 + x = 298$

$500 + x = -6$

$6 + x = 0$

$x + 2000 = 0$

$15 + x = 25$

$15 + x = -25$

$330 + x = 45$

$330 + x = -45$

$-20 + x = 10$

$-20 + x = -10$

$-20 + x = -100$

$0 + x = 15$

$1{,}215{,}687 + x$
$= 1{,}200{,}347$

2. For each of the following equations, give an integer which is a solution. If there is no such integer, say so.

(a) $-3 \cdot x = -21$

(b) $-3 \cdot x = 21$

(c) $-3 \cdot x = 20$

(d) $x \cdot 5 = -50$

(e) $x \cdot 5 = 45$

(f) $x \cdot 5 = 102$

(g) $4 \cdot x = 0$

(h) $0 \cdot x = -2$

(i) $3 \cdot x = 3$

(j) $84 \cdot x = 1$

(k) $1 \cdot x = 84$

(l)) $88 \cdot x = 8000$

(m) $88 \cdot x = 8800$

(n) $x \cdot (-3500) = 0$

(o) $467 \cdot x = -1401$

(p) $-467 \cdot x = 1401$

(q) $467 \cdot x = -1410$

(r) $-12 \cdot x = 144$

(s) $144 \cdot x = -12$

(t) $0 \cdot x = 0$

3. Give the integer for each of the quotients below. If there is no integer, say so.

(a) $-21 \div -3$

(b) $21 \div -3$

(c) $\frac{20}{-3}$

(d) $-50 \div 5$

(e) $45 \div 5$

(f) $\frac{102}{5}$

(g) $0 \div 4$

(h) $\frac{-2}{0}$

(i) $3 \div 3$

(j) $\frac{1}{84}$

(k) $\frac{84}{1}$

(l) $0 \div (-35{,}000)$

(m) $85 \div 5$

(n) $\frac{-42}{14}$

(o) $(-33) \div (-11)$

(p) $\frac{0}{48}$

(q) $(-2000) \div (-1000)$

(r) $\frac{-1000}{2000}$

4. Which of the following statements are *true*? (Be prepared to defend your answers.)

   (a) Addition is a binary operation on the set Z of integers.

   (b) Subtraction is a binary operation on the set Z of integers.

   (c) Multiplication is a binary operation on the set Z of integers.

   (d) Division is a binary operation on the set Z of integers.

5. In the set Z of integers, how many solutions are there to the equation
$$0 \cdot x = 0?$$

   Do you think that "$\frac{0}{0}$" is the name of an integer? Why or why not?

Is "$\frac{0}{5}$" the name of an integer? Why or why not?

Is "$\frac{5}{0}$" the name of an integer? Why or why not?

12.3 Quotients and Ordered Pairs of Integers.

Since 2 is the solution of "$1 \cdot x = 2$," we shall say that 2 is the *quotient* of 2 and 1, and write

$$\frac{2}{1} = 2.$$

In other words, we may use "$\frac{2}{1}$" instead of "2" to represent the number 2. Instead of writing "$1 \cdot 2 = 2$," we can write

$$1 \cdot \frac{2}{1} = 2.$$

Now, $\frac{2}{1}$ is an ordered pair of integers. (It is an ordered pair since it would be incorrect to use "$\frac{1}{2}$" instead of "$\frac{2}{1}$" in the example above.) As we have already noted, $\frac{2}{1}$ is a quotient, namely $2 \div 1$. When written in the form "$\frac{2}{1}$" we shall in this chapter call this quotient a *fraction*.

If x is an integer such that $b \cdot x = a$, then the fraction $\frac{a}{b}$ represents the quotient $a \div b$. The number $\underline{a}$ is the *numerator* of the fraction (or quotient), and the number $\underline{b}$ is the denominator of the fraction (or quotient).

Are there other equations of the type "$b \cdot x = a$" for which the number 2 is a solution? There are in fact many of them. Study the examples below.

2 is the solution of "$1 \cdot x = 2$." So $2 = \frac{2}{1}$.

2 is the solution of "$2 \cdot x = 4$." So $2 = \frac{4}{2}$.

2 is the solution of "$3 \cdot x = 6$." So $2 = \frac{6}{3}$.

2 is the solution of "$4 \cdot 2 = 8$." So $2 = \frac{8}{4}$.

.

.

.

2 is the solution of "$k \cdot x = 2k$." So $2 = \frac{2k}{2}$.

Therefore, any fraction $\frac{2k}{k}$, where k is an integer *not zero*, may be used to represent 2.

Questions Can you explain why we must state that $k \neq 0$ in the above discussion?

$\underline{k}$ may be a negative number, since 2 is a solution, for instance, of "$-3 \cdot x = -6$."

That is, the quotient $-6 \div -3$ is 2. When this quotient is written in the form $\frac{-6}{-3}$, we shall still call it a fraction. (Notice that we allow the numerator or denominator of a fraction to be negative.)

So we see that the number 2 may be represented by a whole set of fractions, indicated as follows:

$$(\ldots \frac{-4}{-2}, \frac{-2}{-1}, \frac{2}{1}, \frac{4}{2}, \frac{6}{3}, \frac{8}{4}, \ldots \frac{2k}{k}, \ldots). \ (k \neq 0)$$

Consider now another integer, say -10. With -10 also we associate an infinite class of fractions (that is, quotients). To see this, note that -10 is the solution of such equations as

$$1 \cdot x = -10, \quad 2 \cdot x = -20, \quad 3 \cdot x = -30.$$

So, -10 may be represented by such fractions as

$$\frac{-10}{1}, \qquad \frac{-20}{2}, \qquad \frac{-30}{3}.$$

And in fact -10 may be represented by the infinite set of fractions indicated below

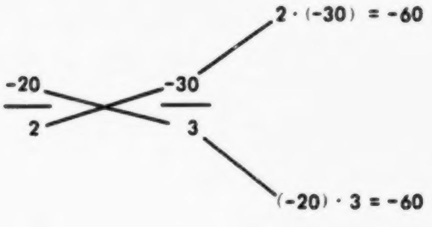
$$(\frac{-10}{1}, \frac{-20}{2}, \frac{-30}{3}, \frac{-40}{4}, \ \ldots, \ \frac{-10k}{k}, \ldots).$$

Of course, again we must say that $k \neq 0$. *k* might be a negative number, however, For instance, $\frac{30}{-3}$, which is the quotient of 30 and -3, may be used to represent -10. That is, -10 is the solution of "$-3 \cdot x = 30$."

Question: Which of the following fractions represents -10:

$$\frac{50}{5}, \frac{50}{-5}, \frac{-100}{10}, \frac{100}{10}, \frac{-100}{10}, \frac{-5000}{500}.$$

Let us select two fractions, $\frac{-20}{2}$ and $\frac{-30}{3}$, from the set of fractions representing -10. Notice that $(-20) \cdot 3 = 2 \cdot (-30)$, since each product is -60. We can say that "the cross-products are equal," a phrase suggested by the diagram below.



$\frac{-20}{2}$ and $\frac{-30}{3}$ furnish an example of what we call *equivalent fractions*. Thus, two fractions $\frac{a}{b}$ and $\frac{c}{d}$, for which $ad = bc$, are equivalent fractions. Furthermore, two equivalent fractions represent the same quotient.

Question: Can show that any two fractions representing -10 are equivalent fractions?

Example Represent the number 5 by an infinite set of fractions.

5 is the solution of "$1 \cdot x = 5$." Therefore, the fraction $\frac{5}{1}$ (the quotient of 5 and 1) may be be used to represent 5. Also, any fraction equivalent to $\frac{5}{1}$ may be used to represent 5. The set is indicated below:

$(\frac{5}{1}, \frac{10}{2}, \frac{15}{3}, \frac{20}{4}, \ldots, \frac{5k}{k}, \ldots)$.

Each of these fractions indicates a quotient. For example, to say that "$5 = \frac{15}{3}$" is to say that 5 is the quotient $15 \div 3$; that is, 5 is the solution of "$3 \cdot x = 15$."

12.4 Exercises.

1. (a) What integer is the solution of the equation "$3 \cdot x = 12$"?
   (b) List *four different fractions* which represent the solution of the equation in part (a).
   (c) Indicate the *entire set* of fractions which represent the solution of the equation in part (a).
2. (a) Indicate the set of fractions representing the integer 8.
   (b) Indicate the set of fractions representing the integer 13.
   (c) Indicate the set of fractions representing the integer -2.
3. Complete each of the following so that a true statement results.

   (a) $5 \cdot \frac{15}{5} =$ (b) $7 \cdot \frac{35}{7} =$ (c) $-3 \cdot \frac{6}{-3} =$

   (d) $100 \cdot \frac{-500}{100} =$ (e) $9 \cdot 4 =$ (f) $9 \cdot \frac{36}{9} =$

   (g) $-5 \cdot 3 =$ (h) $-5 \cdot \frac{-15}{-5} =$
4. Which of the following pairs of fractions are equivalent?

   (a) $\frac{20}{2}, \frac{100}{10}$ (b) $\frac{-15}{3}, \frac{10}{2}$ (c) $\frac{-8}{4}, \frac{10}{-5}$

   (d) $\frac{0}{5}, \frac{0}{9}$ (e) $\frac{6}{6}, \frac{-19}{-19}$ (f) $\frac{18}{3}, \frac{24}{6}$
5. For each pair of fractions below, tell what integer x must be so that the two fractions represent the same quotient.

   (a) $\frac{14}{2}, \frac{x}{5}$ (b) $\frac{12}{3}, \frac{8}{x}$ (c) $\frac{0}{31}, \frac{x}{3}$

   (d) $\frac{45}{45}, \frac{2}{x}$ (e) $\frac{3}{2}, \frac{9}{x}$ (f) $\frac{100}{x}, \frac{30}{3}$

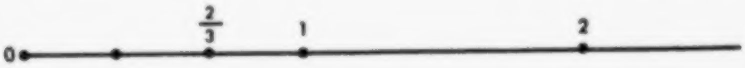
   (g) $\frac{33}{3}, \frac{x}{1}$ (h) $\frac{2k}{k}, \frac{x}{5}$ $(k \neq 0)$
6. (a) Consider the fraction $\frac{3}{1}$, with numerator 3 and denominator 1. If we multiply the numerator by 2 and also multiply the denominator by 2, we get the fraction $\frac{6}{2}$. Are the fractions $\frac{3}{1}$ and $\frac{6}{2}$ equivalent? What integer do they represent?
   (b) If both the numerator and denominator of the fraction $\frac{3}{1}$ are multiplied by 3, what fraction results? Is it equivalent to $\frac{3}{1}$? Why or why not?
   (c) If both numerator and denominator of the fraction $\frac{3}{1}$ are multiplied by -2, is the resulting fraction equivalent to $\frac{3}{1}$? Why or why not?
   (d) If both numerator and denominator of $\frac{3}{1}$ are multiplied by k, where k is some integer not zero, is the resulting fraction equivalent to $\frac{3}{1}$? Why or why not?
   (e) If $k = 0$, are $\frac{3}{1}$ and $\frac{3k}{k}$ equivalent? Why or why not?
7. Consider the equation "$5 \cdot x = 0$."
   (a) What integer is the solution of this equation?
   (b) What integer is represented by the fraction $\frac{0}{5}$?
8. Consider the equation "$-2 \cdot x = 0$."
   (a) What integer is the solution of this equation?
   (b) What integer is represented by the fraction $\frac{0}{-2}$?
9. (a) Are the fractions $\frac{0}{5}$ and $\frac{0}{-2}$ equivalent? Why or why not?
   (b) Indicate the entire set of fractions representing the integer 0.
10. Consider the equation "$0 \cdot x = 5$."
    (a) What integer is the solution of this equation?
    (b) Does the fraction $\frac{5}{0}$ represent an integer?
11. Consider the equation "$0 \cdot x = 0$."
    (a) What is the solution set of this equation?
    (b) Is there one particular integer with which the fraction $\frac{0}{0}$ may be associated?
12. Explain why we cannot allow the quotient $\frac{a}{0}$, where a is an integer.

12.5 Rational Numbers.

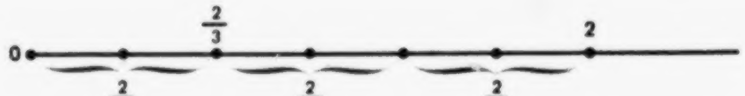
Does the equation

$$3 \cdot x = 2$$

have a solution? Certainly there is no *integer* which is a solution. (Can you give an argument to show that there is no such integer?) However, you may recall the following way to illustrate a meaning of the fraction $\frac{2}{3}$.

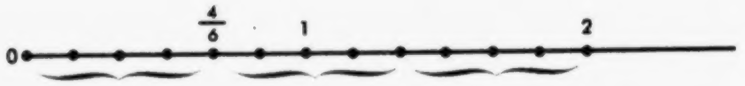


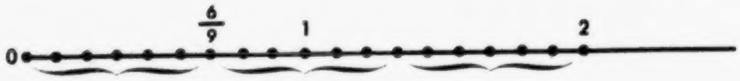
You may also remember that a diagram such as the one below suggests that $3 \cdot \frac{2}{3} = 2$.



And it is just as sensible to agree (as the diagrams below suggest) that

$$3 \cdot \frac{4}{6} = 2 \text{ and } 3 \cdot \frac{6}{9} = 2.$$





Now if we are going to extend the integers so that the equation "$3 \cdot x = 2$" has a solution, we would like *exactly one solution*, not more than one. (Why do we want this? Well, if there were two solutions, then we would have $3 \cdot x = 3 \cdot y$ but $x \neq y$. That is, we would not have a cancellation law in this new system of numbers. But we do not want to destroy the properties of the integers which we already have. And this is why we demand that the equation have one and only one solution.)

We shall agree therefore that the fractions $\frac{2}{3}$, $\frac{4}{6}$, and $\frac{6}{9}$ represent *the same number*, namely the solution of "$3 \cdot x = 2$." In fact we shall agree that any fraction *equivalent* to these fractions represents the same number. Just as in Section 13.3, we take two fractions $\frac{a}{b}$ and $\frac{c}{d}$ to be equivalent if $a \cdot d = b \cdot c$. Hence, we have the following set of fractions for the solution of "$3 \cdot x = 2$":

$$(\cdots, \frac{-4}{-6}, \frac{-2}{-3}, \frac{2}{3}, \frac{4}{6}, \frac{6}{9}, \frac{8}{12}, \frac{10}{15}, \cdots).$$

Notice that we allow numerators and denominators to be negative integers. Thus, the fraction $\frac{-2}{-3}$ is in the set because it is equivalent to $\frac{2}{3}$.

Question: Which of the following fractions are also in the set of fractions representing the solution of "$3 \cdot x = 2$"?

$$\frac{-6}{-9}, \frac{30}{40}, \frac{3}{2}, \frac{24}{36}, \frac{2k}{3k} \ (k \neq 0)$$

Thus, we have a new number which is really an entire set of equivalent fraction: any fraction in the set may be used to represent the number. Such a number is a *rational number*.

A rational number is a set of equivalent fractions.

Also, the rational number $\frac{2}{3}$ arose as the solution of "$3 \cdot x = 2$." And in general we say that a rational number is a solution of an equation "$b \cdot x = a$," where a and b are integers. However, we do not want to destroy our previous results in arithmetic. And, as we saw in Exercises 10 and 11 of Section 13.4, equations such as "$0 \cdot x = a$," where a is an integer, cause trouble. Therefore, we say

> A rational number is a number which is the solution of an equation
>
> $$b \cdot x = a,$$
>
> where a and b are integers, but $b \neq 0$. This number is represented by the fraction $\frac{a}{b}$, or by any fraction equivalent to it. Thus, we have $b \cdot \frac{a}{b} = a$.

Do you see from this definition that the *denominator of a fraction is never zero?* That is, a fraction with zero denominator does not represent a rational number.

Example 1. Solve the equation "$3 \cdot x = 4$." The solution of this equation is the rational number represented by the following set of equivalent fractions:

$$(\cdots, \frac{-4}{-3}, \frac{4}{3}, \frac{8}{6}, \frac{12}{9}, \frac{16}{12}, \cdots)$$

Once again we see that a rational number is a set of equivalent fractions. We do not, of course, write all of these fractions when we want to refer to the number. We simply choose one of them and say, for instance, "the rational number $\frac{4}{3}$," and this means the rational number to which the fraction $\frac{4}{3}$ belongs. Of course, ''the rational number $\frac{8}{6}$'' refers to exactly the same number; and this is what we mean when we write

$$\frac{4}{3} = \frac{8}{6}$$

as a statement about rational numbers.

Example 2. What is the solution of "$2 \cdot x = 5$"? The solution is the rational number $\frac{5}{2}$; that is $2 \cdot \frac{5}{2} = 5$. It is also correct to say that the solution is $\frac{10}{4}$, and $2 \cdot \frac{10}{4} = 5$. In fact, any fraction in the following set may be used to represent the solution:

$$(\ldots, \frac{-5}{-2}, \frac{5}{2}, \frac{10}{4}, \frac{15}{6}, \frac{20}{8}, \frac{25}{10}, \ldots)$$

To represent the rational number of Example 2 fraction $\frac{5}{2}$ is often used. This is because it has a positive denominator, and its numerator and denominator have no common factor other than 1. Such a fraction is called an *irreducible fraction.*

Questions: $\frac{5}{21}$ is an irreducible fraction. Why?

$\frac{6}{21}$ is not an irreducible fraction. Why not?

There is still another way to describe a rational number. In (Z, ·) we use a multiplication fact such as

$$3 \cdot 4 = 12$$

to define the division fact

$$12 \div 3 = 4.$$

And we shall continue to define division in this way for rational numbers. Therefore, from the multiplication fact

$$3 \cdot \frac{2}{3} = 2,$$

we get the division fact

$$2 \div 3 = \frac{2}{3}.$$

In this way, the rational number $\frac{2}{3}$ is the quotient of two integers. And, in general, we say

> A rational number $\frac{a}{b}$ is the quotient $a \div b$ of the the integers a and b. ($b \neq 0$)

Thus, we are still able to say that a fraction $\frac{a}{b}$ is a *quotient*, even when that quotient is not an integer.

12.6 Exercises.

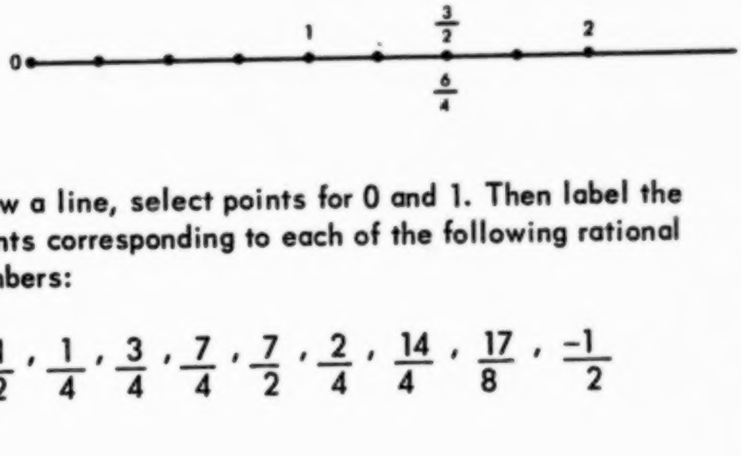
1. Below are a number of equations, each of which has a solution which is a rational number. For each equation, write the irreducible fraction which represents the solution. Then write four other fractions for the number.

(a) $7 \cdot x = 5$
(b) $15 \cdot x = 10$
(c) $4 \cdot x = 1$
(d) $10 \cdot x = 1$
(e) $5 \cdot x = 2$
(f) $10 \cdot x = 4$
(g) $-3 \cdot x = 2$
(h) $3 \cdot x = -2$
(i) $b \cdot x = a$ ($b \neq 0$)

2. Complete the following so as to make a true statement:

(a) $5 \cdot 3/5 =$
(b) $7 \cdot 2/7 =$
(c) $3 \cdot 10/3 =$
(d) $412 \cdot 27/412$
(e) $-5 \cdot 3/-5 =$
(f) $7 \cdot -11/7 =$
(g) $17 \cdot 29/17 =$
(h) $b \cdot a/b =$

3. You are already familiar with coordinates of a line, and even with rational numbers as coordinates of a line. For example on the line below the fraction 3/2 has been used to determine a point of the line. Also, the fraction 6/4 has been used to determine a point. And they determine the same point. But this is as it should be, for we have already agreed that the fractions 3/2 and 6/4 denote the same number.



Draw a line, select points for 0 and 1. Then label the points corresponding to each of the following rational numbers:

$$\frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \frac{7}{4}, \frac{7}{2}, \frac{2}{4}, \frac{14}{4}, \frac{17}{8}, \frac{-1}{2}$$

4. Complete the following so that a true statement results; that is, so that the two fractions represent the same rational number. The example has been done correctly.

$\frac{3}{4} = \frac{x}{8}$ If $3 \cdot 8 = 4 \cdot x$, then $x = 6$.

So, $\frac{3}{4} = \frac{6}{8}.$

(a) $\frac{30}{12} = \frac{5}{x}$
(b) $\frac{x}{3} = \frac{56}{21}$
(c) $\frac{0}{25} = \frac{x}{4}$

(d) $\frac{0}{25} = \frac{0}{x}$

(e) $\frac{48}{x} = \frac{12}{3}$

(f) $\frac{15}{9} = \frac{x}{6}$

5. For each of the rational numbers below, write two different equations of which the number is a solution.

(a) $\frac{7}{3}$ (c) $\frac{4}{8}$ (e) $\frac{100}{3}$ (g) $\frac{0}{5}$

(b) $\frac{2}{9}$ (d) $\frac{15}{4}$ (f) $\frac{36}{24}$ (h) $\frac{6}{2}$

6. We have said that two fractions a/b and c/d are equivalent if ad = bd.
   - (a) Are the fractions 7/13 and 91/169 equivalent?
   - (b) What are the three properties which an equivalence relation must have? (See Section 8.11)
   - (c) Show that any fraction is equivalent to itself (the reflexive property)
   - (d) Give an argument showing that if a/b is equivalent to c/d then c/d is equivalent to a/b (symmetry).
   - (e) Give an argument showing that if a/b is equivalent to c/d, and c/d is equivalent to e/f, then a/b is equivalent to e/f.
   - (f) Show that if we admitted a fraction such as 0/0, then it would be equivalent to every fraction.

7. (a) In the set of integers, what is the solution of $1 \cdot x = 5$?
   (b) In the set of rational numbers, what is the solution of $1 \cdot x = 5$? (The answers to these questions are not the same!)

12.7 Multiplication of Rational Numbers.

Given the equation

$$3 \cdot x = 6,$$

What is the solution?

There are really two ways to answer. In the system (Z,·), the solution is certainly the integer 2. In the new set of rational numbers which we are developing the solution is the rational number represented by any fraction in the set

$$\left\{ \frac{2}{1}, \frac{4}{2}, \frac{6}{3}, \frac{8}{4}, \frac{10}{5}, \ldots \right\}$$

So, as we have already noticed in some of the exercises, there is a very strong connection between the integer 2 and the rational number $\frac{2}{1}$. We shall keep this connection in mind as we learn to multiply rational numbers.

Consider now the two equations

$$3 \cdot x = 6 \text{ and } 2 \cdot y = 10.$$

Each of them has an integer as a solution; in order to make the sentences true, x must be 2, and y must be 5. Furthermore, we know that in (Z,·) the product of 2 and 5 is 10. Now if we think instead of rational numbers, the solutions of the above equations may be represented by the fractions

$$\frac{6}{3} \text{ and } \frac{10}{2}.$$

We would like for the product of these rational numbers to be the rational number $\frac{10}{1}$. Recalling the way you learned to multiply fractions in elementary school, we have

$$\frac{6}{3} \cdot \frac{10}{2} = \frac{6 \cdot 10}{3 \cdot 2}$$

$$= \frac{60}{6}$$

And the fraction $\frac{60}{6}$ does represent the rational number $\frac{10}{1}$. (Why?)

It would seem a good idea then to adopt this method as a way to multiply rational numbers. There is one question, however, since every rational number has an infinite number of fractions which represent it. Which fraction do you choose when you are finding a product? The following examples will suggest an answer to this question.

Example 1. What is the product of the rational numbers $\frac{2}{3}$ and $\frac{5}{7}$?

$$\frac{2}{3} \cdot \frac{5}{7} = \frac{2 \cdot 5}{3 \cdot 7}$$

$$= \frac{10}{21}$$

Now the rational number $\frac{2}{3}$ may be represented by any fraction in the set

$$\left\{ \frac{2}{3}, \frac{4}{6}, \frac{6}{9}, \frac{8}{12}, \ldots \right\},$$

and the rational number $\frac{5}{7}$ may be represented by any fraction in the set

$$\left\{ \frac{5}{7}, \frac{10}{14}, \frac{15}{21}, \frac{20}{28}, \ldots \right\}.$$

How would the product be affected if, in finding the product of the rational numbers $\frac{2}{3}$ and $\frac{5}{7}$, we used fractions other than those used in Example 1?

Example 2. Find the product of the rational numbers $\frac{2}{3}$ and $\frac{5}{7}$. (Note that this is the same as Example 1.)

The fraction $\frac{6}{9}$ represents the rational number $\frac{2}{3}$.

The fraction $\frac{10}{14}$ represents the rational number $\frac{5}{7}$.

$$\frac{6}{9}\cdot\frac{10}{14}=\frac{6\cdot 10}{9\cdot 14}$$

$$=\frac{60}{126}.$$

Are the results in Example 1 and 2 the same? They are, since the fractions $\frac{10}{21}$ and $\frac{60}{126}$ represent the same rational number. (Why?)

As a matter of fact, although we do not prove it now, it is true that you may use any fractions representing two rational numbers when you are finding their product. That is, for any ordered pair of rational numbers, the operation of multiplication assigns *one and only one* rational number, regardless of the fractions used to represent them.

We now make the following definition:

> If $\frac{a}{b}$ and $\frac{c}{d}$ are fractions representing two rational numbers, then the fraction $\frac{ac}{bd}$ represents the product of these numbers.

We have stated this definition in terms of fractions in order to emphasize that you may use any fractions representing the numbers. Often, however, the definition is given in the following way:

> If $\frac{a}{b}$ and $\frac{c}{d}$ are rational numbers, $\frac{a}{b}\cdot\frac{c}{d}=\frac{ac}{bd}$.

12.8 Exercises.

1. Find the following products of rational numbers. Use an *irreducible fraction* to denote each answer.

(a) $\frac{3}{5}\cdot\frac{2}{9}$

(b) $\frac{5}{8}\cdot\frac{3}{7}$

(c) $\frac{10}{11}\cdot\frac{4}{5}$

(d) $\frac{4}{5}\cdot\frac{10}{11}$

(e) $\frac{10}{3}\cdot\frac{3}{1}$

(f) $\frac{7}{2}\cdot\frac{3}{5}$

(g) $\frac{3}{5}\cdot\frac{7}{2}$

(h) $\frac{4}{9}\cdot\frac{2}{7}$

(i) $\frac{2}{7}\cdot\frac{4}{9}$

(j) $\frac{0}{3}\cdot\frac{5}{8}$

(k) $\left(\frac{2}{3}\cdot\frac{4}{5}\right)\cdot\frac{7}{6}$

(l) $\frac{2}{3}\cdot\left(\frac{4}{5}\cdot\frac{7}{6}\right)$

(m) $\frac{0}{3}\cdot\left(\frac{6}{6}\cdot\frac{3}{2}\right)$

(n) $\frac{5}{8}\cdot\left(\frac{21}{2}\cdot\frac{4}{9}\right)$

(o) $\left(\frac{5}{8}\cdot\frac{21}{2}\right)\cdot\frac{4}{9}$

2. Find each of the following products. Some of the products are products of integers, while others are products of rational numbers.

(a) $5\cdot 2$

(b) $\frac{5}{1}\cdot\frac{2}{1}$

(c) $3\cdot 6$

(d) $\frac{3}{1}\cdot\frac{6}{1}$

(e) $7\cdot 8$

(f) $\frac{7}{1}\cdot\frac{8}{1}$

(g) $15\cdot 5$

(h) $\frac{15}{1}\cdot\frac{5}{1}$

(i) $-2\cdot 3$

(j) $\frac{-2}{1}\cdot\frac{3}{1}$

(k) $(-4)(-6)$

(l) $\frac{-4}{1}\cdot\frac{-6}{1}$

*3. On the basis of the products in Exercise 2, can you give an argument that the system $(Z,\cdot)$ is *isomorphic* to the system composed of certain rational numbers and multiplication.

4. Determine the following products of rational numbers. Represent the product by an irreducible fraction.

(a) $\frac{3}{3}\cdot\frac{2}{5}$

(b) $\frac{4}{3}\cdot\frac{7}{7}$

(c) $\frac{5}{6}\cdot\frac{1}{1}$

(d) $\frac{10}{7}\cdot\frac{8}{8}$

(e) $\frac{1000}{1000}\cdot\frac{9}{5}$

(f) $\frac{6}{6}\cdot\frac{3}{4}$

5. Determine the following products. Use an irreducible fraction to represent each product.

(a) $\frac{2}{3}\cdot\frac{3}{2}$

(b) $\frac{5}{7}\cdot\frac{7}{5}$

(c) $\frac{9}{4}\cdot\frac{4}{9}$

(d) $\frac{10}{3}\cdot\frac{3}{10}$

(e) $\frac{100}{7}\cdot\frac{7}{100}$

(f) $\frac{1}{1}\cdot\frac{1}{1}$

(g) $\frac{22}{5}\cdot\frac{5}{22}$

(h) $\frac{14}{99}\cdot\frac{99}{14}$

12.9 Properties of Multiplication.

We shall use Q to name the set of rational numbers. And with the introduction of the binary operation multiplication, we have the operational system.

$$(Q,\cdot).$$

As with all operational systems, it is worthwhile to investigate the properties of $(Q,\cdot)$. As you probably recognized from Exercise 1 of Section 13.8, multiplication of rational numbers is both commutative and associative.

Commutative Property of $(Q,\cdot)$

If $\frac{a}{b}$ and $\frac{c}{d}$ are rational numbers, then

$$\frac{a}{b}\cdot\frac{c}{d}=\frac{c}{d}\cdot\frac{a}{b}$$

Associative Property of (Q, ·)

If $\frac{a}{b}, \frac{c}{d}$, and $\frac{e}{f}$ are rational numbers, then

$$\left(\frac{a}{b} \cdot \frac{c}{d}\right) \cdot \frac{e}{f} = \frac{a}{b} \cdot \left(\frac{c}{d} \cdot \frac{e}{f}\right)$$

If you refer to Exercise 4 of Section 13.8, you should ee that there is an identity element in (Q,·). This dentity element is the rational number associated with he following set of fractions:

$$\left\{ \frac{1}{1}, \frac{2}{2}, \frac{3}{3}, \frac{4}{4}, \ldots \right\}$$

Example 1. $\frac{3}{4} \cdot \frac{2}{2} = \frac{6}{8}$

$= \frac{3}{4}.$

Example 2. $\frac{3}{4} \cdot \frac{5}{5} = \frac{15}{20}$

$= \frac{3}{4}$

Examples 1 and 2 are really the same rational number products. In both cases, the rational number $\frac{3}{4}$ was multiplied by the same rational number; the only difference is that in the first example the fraction $\frac{2}{2}$ was used to represent the number, while in the second example the fraction $\frac{5}{5}$ was used. But in both cases the product was $\frac{3}{4}$ since the fractions $\frac{2}{2}$ and $\frac{5}{5}$ represent the identity element of (Q,·). Since $\frac{1}{1}$ is the irreducible fraction representing this number, we write

Identity Element of (Q,·)

If $\frac{a}{b}$ is a rational number, then $\frac{a}{b} \cdot \frac{1}{1} = \frac{a}{b}$

What is the product of $\frac{2}{3}$ and $\frac{3}{2}$? It is easy to check that the product is $\frac{1}{1}$, the identity element of (Q,·); therefore, these rational numbers are inverses of each other in this system. If you refer to Exercise 5 of Section 3.8, you should notice a pattern—the inverse of $\frac{a}{b}$ is $\frac{b}{a}$. There is one important exception to this rule however. The product of $\frac{0}{1}$ and another rational number cannot be $\frac{1}{1}$.

Question: If $\frac{a}{b}$ is any rational number, what is the product $\frac{0}{1} \cdot \frac{a}{b}$? Do you see then why $\frac{0}{1}$ has no inverse in (Q,·)?

We now state the following property:

Inverse Property of (Q,·)

If $\frac{a}{b}$ is a rational number which is not $\frac{0}{1}$ (that is, $a \neq 0$), then $\frac{b}{a}$ is the inverse of $\frac{a}{b}$, i.e., $\frac{a}{b} \cdot \frac{b}{a} = \frac{1}{1}$

Thus, the rational numbers $\frac{a}{b}$ and $\frac{b}{a}$ are inverses in (Q,·). Because the operation in this system is multiplication, we may call them *multiplicative inverses*. It is also common in the system (Q,·) to call a multiplicative inverse a *reciprocal*.

Example 3. The multiplicative inverse of the number $\frac{2}{3}$ is $\frac{3}{2}$, or the reciprocal of $\frac{2}{3}$ is $\frac{3}{2}$.

12.10 Exercises.

1. For each of the following equations, find the solution in (Q,·).

(a) $\frac{2}{3} \cdot a = \frac{2}{3}$

(b) $\frac{4}{3} \cdot a = \frac{1}{1}$

(c) $\frac{10}{9} \cdot \frac{9}{10} = x$

(d) $\frac{3}{3} \cdot x = \frac{5}{6}$

(e) $\frac{4}{5} \cdot m = \frac{1}{1}$

(f) $\frac{10}{7} \cdot a = \frac{1}{1}$

(g) $\frac{15}{4} \cdot x = \frac{15}{4}$

(h) $x \cdot x = \frac{1}{1}$

2. Determine each of the following products:

(a) $\frac{0}{5} \cdot \frac{2}{3}$

(b) $\frac{5}{9} \cdot \frac{0}{1}$

(c) $\frac{23}{11} \cdot \frac{0}{1}$

(d) $\frac{0}{8} \cdot \frac{4}{5}$

(e) $\frac{0}{17} \cdot \frac{35}{8}$

(f) $\frac{0}{1} \cdot \frac{1}{1}$

(g) $\frac{0}{1} \cdot \frac{0}{1}$

3. The rational number $\frac{0}{1}$ is represented by any one of the fractions in the set:

$$\left\{\cdots, \frac{0}{-2}, \frac{0}{-1}, \frac{0}{1}, \frac{0}{2}, \frac{0}{3}, \cdots\right\}.$$

On the basis of the products in problem 2, how would you describe the behavior of this number in multiplication?

4. (a) Express the identity element of (Q,·) as a set of equivalent fractions.
   (b) Express the inverse of the rational number $\frac{2}{5}$ as a set of equivalent fractions.

(c) What is the product of $\frac{3}{4}\cdot\frac{8}{6}$?

(d) What rational number is its own inverse in the system (Q,·)?

(e) What rational number has no inverse in the system (Q,·)?

5. (a) Write the properties which a system (S,*) must have in order to be a *group*.

(b) Is (Z,·) a group? If so, is it commutative?

(c) Is (Q,·) a group? If so, is it commutative?

(d) Let X be the set of all rational numbers except $\frac{0}{1}$. Is (X,·) a group? If so, is it commutative?

6. (a) Compute the following products in (Z,·)

$-8\cdot 1=\qquad 14\cdot 1=\qquad -234\cdot 1=\qquad 55\cdot 1=$

$86\cdot 0=\qquad -14\cdot 0=$

(b) Compute the following products in (Q,·)

$\frac{-8}{1}\cdot\frac{1}{1}=\qquad \frac{14}{1}\cdot\frac{1}{1}=\qquad \frac{-234}{1}\cdot\frac{1}{1}=\qquad \frac{55}{1}\cdot\frac{1}{1}=$

$\frac{86}{1}\cdot\frac{0}{1}=\qquad \frac{-14}{1}\cdot\qquad \frac{0}{1}=$

7. Often a short cut can be used in finding the product of two rational numbers. Perhaps you have used this short cut before, but have never been able to explain why it "works."
Study the following example:

$$\frac{2}{3}\cdot\frac{5}{6}=\frac{2\cdot 5}{3\quad 6}=\frac{2\cdot 5}{3\cdot(2\cdot 3)}=\frac{2\cdot\cdot 5}{2\cdot(3\cdot 3)}=\frac{2}{2}\cdot\frac{5}{3\cdot 3}=\frac{5}{9}$$

This is not a short cut! But notice that since $\frac{2}{2}$ is the identity element of multiplication, we could have determined the product this way:

$$\frac{\overset{1}{\cancel{2}}}{3}\cdot\frac{5}{\underset{3}{\cancel{6}}}=\frac{5}{9}$$

Do you see how the identity element of multiplication has been used in the following example?

$$\frac{\overset{2}{\cancel{4}}}{\underset{\propto}{\cancel{15}}}\cdot\frac{\overset{3}{\cancel{9}}}{\underset{5}{\cancel{10}}}=\frac{6}{25}$$

Use this short cut in finding the following products:

(a) $\frac{7}{3}\cdot\frac{5}{14}$

(b) $\frac{5}{8}\cdot\frac{7}{5}$

(c) $\frac{10}{9}\cdot\frac{27}{2}$

(d) $\frac{5}{2}\cdot\frac{2}{5}$

(e) $\frac{18}{45}\cdot\frac{15}{27}$

(f) $\frac{5}{4}\cdot\left(\frac{24}{7}\cdot\frac{7}{36}\right)$

(g) $\left(\frac{3}{3}\cdot\frac{9}{10}\right)\cdot\frac{15}{27}$

(h) $\left(\frac{24}{11}\cdot\frac{33}{42}\right)\cdot\frac{1}{2}$

(i) $\frac{2}{3}\cdot\frac{3}{4}\cdot\frac{4}{5}\cdot\frac{5}{6}\cdot\frac{6}{7}$

(j) $\frac{a}{b}\cdot\frac{b}{c}\cdot\frac{c}{d}\cdot\frac{e}{f}$

12.11 Division of Rational Numbers.

In (Z, ·), the equation

$$12\div 3=x$$

has the solution 4, because $4\cdot 3=12$. That is, division is defined in terms of multiplication. We want to define division this way also in (Q,·). Suppose then we have the equation

$$\frac{3}{4}\div\frac{2}{5}=\frac{x}{y}$$

Is there a solution? If there is, we want the following to be true:

$$\frac{x}{y}\cdot\frac{2}{5}=\frac{3}{4}.$$

Now the reciprocal of $\frac{2}{5}$ is $\frac{5}{2}$. And we know that

$$\frac{5}{2}\cdot\frac{2}{5}=\frac{1}{1}.$$

therefore,

$$\frac{3}{4}\cdot\left(\frac{5}{2}\cdot\frac{2}{5}\right)=\frac{3}{4}$$

And, using the associative property of multiplication, we can write

$$\left(\frac{3}{4}\cdot\frac{5}{2}\right)\cdot\frac{2}{5}=\frac{3}{4}.$$

Do you see that we have found the number $\frac{x}{y}$ which we were trying to find? It is the product $\frac{3}{4}\cdot\frac{5}{2}$, which is the rational number $\frac{15}{8}$.

So, $\frac{3}{4}\cdot\frac{5}{2}$ is the solution of $\frac{3}{4}\div\frac{2}{5}=\frac{x}{y}$. In other words,

$$\frac{3}{4}\div\frac{2}{5}=\frac{3}{4}\cdot\frac{5}{2}.$$

From this one example, it would seem that the *quotient* of two rational numbers can be found by finding the *product* of two rational numbers.

See if you can follow the steps in the following example:

$$\frac{4}{3}\div\frac{3}{2}=\frac{x}{y}$$

$$\frac{x}{y}\cdot\frac{3}{2}=\frac{4}{3}$$

Now, $\frac{2}{3}\cdot\frac{3}{2}=\frac{1}{1}$

So, $\frac{4}{3}\cdot\left(\frac{2}{3}\cdot\frac{3}{2}\right)=\frac{4}{3}$

$$\left(\frac{4}{3}\cdot\frac{2}{3}\right)\cdot\frac{3}{2}=\frac{4}{3}.$$

So we have found the rational number whose product with $\frac{3}{2}$ is $\frac{4}{3}$; and that is the number $\frac{x}{y}$ which we were seeking. Therefore,

$$\frac{4}{3} \div \frac{3}{2} = \frac{4}{3} \cdot \frac{2}{3} \text{ (which of course is } \frac{8}{9}\text{)}$$

Thus, instead of dividing by $\frac{3}{2}$, you may *multiply* by the *reciprocal* of $\frac{3}{2}$. And if you look at the first example again, you see the same pattern there: instead of dividing by $\frac{2}{5}$, you may multiply by the reciprocal of $\frac{2}{5}$.

Finally, let $\frac{a}{b}$ and $\frac{c}{d}$ be two rational numbers. $(c \neq 0)$

If $\frac{a}{b} \div \frac{c}{d} = \frac{x}{y}$, then $\frac{x}{y} \cdot \frac{c}{d} = \frac{a}{b}$

But we know $(\frac{a}{b} \cdot \frac{d}{c}) \cdot \frac{c}{d} = \frac{a}{b}$ (Why? Can you supply the missing step?)

So $\frac{x}{y} = \frac{a}{b} \cdot \frac{d}{c}$. That is,

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}.$$

Can you complete the following sentence?

Dividing by the rational number, $\frac{x}{y}$ is equivalent to multiplying by______________.

12.12 Exercises.

1. Find the following quotients of rational numbers. Then use a product to show that your result is correct.

(a) $\frac{3}{8} \div \frac{1}{2}$

(b) $\frac{1}{2} \div \frac{3}{8}$

(c) $\frac{5}{4} \div \frac{2}{3}$

(d) $\frac{2}{3} \div \frac{5}{4}$

(e) $\frac{7}{10} \div \frac{1}{12}$

(f) $\frac{1}{12} \div \frac{7}{10}$

2. Find the following quotients of rational numbers.

(a) $\frac{14}{5} \div \frac{3}{7}$

(b) $\frac{8}{9} \div \frac{9}{8}$

(c) $\frac{8}{9} \div \frac{8}{9}$

(d) $\frac{0}{3} \div \frac{2}{5}$

(e) $\frac{4}{11} \div \frac{11}{11}$

(f) $\frac{4}{9} \div \frac{1}{2}$

(g) $\frac{4}{9} \div \frac{1}{3}$

(h) $\frac{4}{9} \div \frac{1}{4}$

(i) $(\frac{5}{4} \div \frac{1}{2}) \div \frac{2}{3}$

(j) $\frac{5}{4} \div (\frac{1}{2} \div \frac{2}{3})$

(k) $\frac{7}{3} \div (\frac{1}{3} \div \frac{2}{1})$

(l) $(\frac{7}{3} \div \frac{1}{3}) \div \frac{2}{1}$

3. Find the following quotients. Some are quotients of integers; others are quotients of rational numbers.

(a) $6 \div 2$

(b) $\frac{6}{1} \div \frac{2}{1}$

(c) $\frac{12}{2} \div \frac{8}{4}$

(d) $20 \div 5$

(e) $\frac{20}{1} \div \frac{5}{1}$

(f) $5 \div 5$

(g) $\frac{5}{1} \div \frac{5}{1}$

(h) $4 \div 8$

(i) $\frac{4}{1} \div \frac{8}{1}$

4. Determine the rational number solution of each of the following equations.

(a) $\frac{2}{3} \cdot \frac{x}{y} = \frac{3}{4}$

(b) $\frac{3}{4} \cdot \frac{x}{y} = \frac{2}{3}$

(c) $\frac{x}{y} \cdot \frac{2}{3} = \frac{5}{7}$

(d) $\frac{5}{7} \cdot \frac{2}{3} = \frac{x}{y}$

(e) $\frac{5}{6} \cdot \frac{x}{y} = \frac{4}{5}$

(f) $\frac{2}{3} \div \frac{x}{y} = \frac{4}{5}$

(g) $\frac{4}{9} \div \frac{x}{y} = \frac{2}{3}$

(h) $\frac{x}{y} \div \frac{2}{3} = \frac{7}{12}$

(i) $\frac{x}{y} \div \frac{14}{27} = \frac{0}{1}$

(j) $\frac{3}{2} \cdot \frac{x}{y} = \frac{1}{1}$

5. (a) Is it possible to find the following quotient: $\frac{2}{3} \div \frac{0}{1}$? Explain why or why not.

(b) What rational number has no reciprocal? (Indicate this rational number by a set of equivalent fractions.)

(c) In the sentence $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}$, what number must $\frac{c}{d}$ *not* be? Why?

(d) Is division an operation on the rational numbers? Why or why not?

(e) If the number $\frac{0}{1}$ is removed from the set Q of rational numbers, is division an operation on the the set of numbers that remain?

(f) Is division associative? (See Exercise 2)

12.13 Addition of Rational Numbers.

We have already seen the close connection between integers such as 2 and 3 and rational numbers such as d $\frac{3}{1}$. And since in $(Z,+)$, $2+3=5$, it would be rable to have

$$\frac{2}{1}+\frac{3}{1}=\frac{5}{1}$$

ny definition we agree to for addition of rational num- . Of course, it should not make any difference which e many available fractions are used to represent the nal numbers $\frac{2}{1}$ and $\frac{3}{1}$. This suggests for example following:

$$\frac{4}{2}+\frac{6}{2}=\frac{10}{2}\;;\;\frac{6}{3}+\frac{9}{3}=\frac{15}{3}.$$

this in turn suggests that we define addition of ra- al numbers in the following way:

$$\frac{a}{b}+\frac{c}{b}=\frac{a+c}{b}$$

t is, in determining a sum, we select fractions which e the same denominator.

Example 1. What is the sum of the rational numbers $\frac{5}{3}$ and $\frac{2}{3}$?

$$\frac{5}{3}+\frac{2}{3}=\frac{5+2}{3}=\frac{7}{3}.$$

Example 2. What is the sum of the rational numbers $\frac{2}{3}$ and $\frac{3}{4}$?

We may indicate the sum this way: $\frac{2}{3}+\frac{3}{4}$.

However, in order to use the method above, we must find other fractions for these numbers, fractions with the same denominator. Now, the least common multiple of 3 and 4 is 12. So we say that 12 is the *least common denominator* of the denominators 3 and 4. We then represent each of the rational numbers by a fraction with denominator 12.

$$\frac{2}{3}+\frac{3}{4}=\frac{8}{12}+\frac{9}{12}=\frac{17}{12}.$$

Although we do not prove it here, it is true that there one and only one rational number which is the sum of o given rational numbers. For instance, in Example 2, could have used the fractions $\frac{16}{24}$ and $\frac{18}{24}$. (Why?) Then the sum would have been the number represented by the fraction $\frac{34}{24}$. But this is the same as the number $\frac{17}{12}$. (Why?)

In order to get a general definition from the method we have been using, let $\frac{a}{b}$ and $\frac{c}{d}$ be two rational numbers. Then to find the sum $\frac{a}{b}+\frac{c}{d}$, we need to select two fractions that have the same denominator. Do you see that

$$\frac{a}{b}=\frac{ad}{bd} \text{ and } \frac{c}{d}=\frac{bc}{bd}\ ?$$

Thus, we have

$$\begin{aligned}\frac{a}{b}+\frac{c}{d}&=\frac{ad}{bd}+\frac{bc}{bd}\\&=\frac{ad+bc}{bd}\end{aligned}$$

We now have an operational system $(Q,+)$. In this system there are the following properties:

Commutative Property of Addition.
If $\frac{a}{b}$ and $\frac{c}{d}$ are rational numbers, $\frac{a}{b}+\frac{c}{d}=\frac{c}{d}+\frac{a}{b}$.

Associative Property of Addition.
If $\frac{a}{b}$, $\frac{c}{d}$, and $\frac{e}{f}$ are rational numbers,

$$\left(\frac{a}{b}+\frac{c}{d}\right)+\frac{e}{f}=\frac{a}{b}+\left(\frac{c}{d}+\frac{e}{f}\right).$$

Although we do not prove these properties here, there are examples of each of them in the exercises.

Now consider the rational number $\frac{0}{1}$, associated with

$$\left\{\frac{0}{1},\frac{0}{2},\frac{0}{3},\frac{0}{4},\ldots\right\}$$

What are the following sums:

$$\frac{2}{3}+\frac{0}{3}\;;\qquad\frac{5}{6}+\frac{0}{6}\;;\qquad\frac{-2}{7}+\frac{0}{7}\;;\qquad\frac{3}{4}+\frac{0}{1}\ ?$$

We have in fact, for any rational number $\frac{a}{b}$, $\frac{a}{b}+\frac{0}{b}=\frac{a+0}{b}=\frac{a}{b}$. We recognize here the familiar pattern for an identity element; and since the fraction $\frac{0}{b}$ represents the rational number $\frac{0}{1}$ we have the following property:

Identity Element for Addition.
For any rational number $\frac{a}{b}$, $\frac{a}{b}+\frac{0}{1}=\frac{a}{b}$.

In investigating operational systems in the past, the notion of inverse has been tied closely to that of identity element; for two elements are inverses of each other if together they produce the identity element. In this connection, study the following examples:

$$\frac{3}{4} + \frac{-3}{4} = \frac{3 + (-3)}{4} = \frac{0}{4} \qquad \frac{-5}{6} + \frac{5}{6} = \frac{-5+5}{6} = \frac{0}{6}.$$

These and similar examples should make the following property clear:

Inverse Elements of Addition.

If $\frac{a}{b}$ is a rational number, then $\frac{a}{b} + \frac{-a}{b} = \frac{0}{1}$.

($-a$ is the additive inverse of a in the set Z of integers.)
That is, every rational number $\frac{a}{b}$ has an inverse, $\frac{-a}{b}$.

Example 3. What is the inverse of $\frac{6}{5}$ in (Q,+)?

The inverse is $\frac{-6}{5}$; $\frac{6}{5} + \frac{-6}{5} = \frac{0}{1}$.

Example 4. What is the inverse of $\frac{-3}{4}$?

In Z, the additive inverse of -3 is 3; that is, $-(-3) = 3$. So the additive inverse of $\frac{-3}{4}$ in Q is $\frac{3}{4}$; $\frac{-3}{4} + \frac{3}{4} = \frac{0}{1}$.

12.14 Exercises.

1. Find the following sums of rational numbers.

(a) $\frac{1}{2} + \frac{1}{3}$

(b) $\frac{2}{3} + \frac{3}{2}$

(c) $\frac{5}{6} + \frac{-2}{6}$

(d) $\frac{10}{7} + \frac{-3}{2}$

(e) $\frac{14}{9} + \frac{5}{3}$

(f) $\frac{20}{9} + \frac{5}{12}$

(g) $\frac{20}{9} + \frac{-5}{12}$

(h) $\frac{-7}{12} + \frac{13}{16}$

(i) $\frac{3}{4} + \frac{5}{-6}$

(Hint: $\frac{-5}{6}$ represents the same rational number as $\frac{5}{-6}$.)

(j) $\frac{x}{y} + \frac{w}{z}$

2. What rational number is assigned to each of the following ordered pairs by the operation of addition?

(a) $\left(\frac{2}{5}, \frac{3}{10}\right)$

(b) $\left(\frac{3}{10}, \frac{2}{5}\right)$

(c) $\left(\frac{7}{8}, \frac{-3}{20}\right)$

(d) $\left(\frac{-3}{20}, \frac{7}{8}\right)$

(e) $\left(\frac{5}{6}, \frac{3}{32}\right)$

(f) $\left(\frac{3}{32}, \frac{5}{6}\right)$

(g) $\left(\frac{-1}{13}, \frac{1}{17}\right)$

(h) $\left(\frac{1}{17}, \frac{-1}{13}\right)$

3. What property of (Q,+) do the sums in Exercise 2 illustrate?

4. Compute the following:

(a) $\left(\frac{2}{5} + \frac{1}{3}\right) + \frac{3}{2}$

(b) $\frac{2}{5} + \left(\frac{1}{3} + \frac{3}{2}\right)$

(c) $\left(\frac{-3}{4} + \frac{5}{6}\right) + \frac{3}{8}$

(d) $\frac{-3}{4} + \left(\frac{5}{6} + \frac{3}{8}\right)$

5. What property of (Q,+) do the sums in Exercise 4 illustrate?

6. List ten different fractions which represent the number which is the identity element in (Q,+).

7. Compute the following:

(a) $\frac{8}{3} + \frac{-8}{3}$

(b) $\frac{8}{3} + \frac{-16}{6}$

(c) $\frac{9}{11} + \frac{0}{1}$

(d) $\frac{9}{11} + \frac{0}{11}$

(e) $\frac{-3}{14} + \frac{3}{14}$

(f) $\frac{148}{3} + \frac{-148}{3}$

(g) $\frac{148}{3} + \frac{148}{-3}$

(h) $\frac{-81}{7} + \frac{0}{51}$

8. Compute the following sums. Some of them concern integers; in this case, be sure you give the sum as an integer. Others concern rational numbers; in this case, be sure to give the sum as a rational number.

(a) $7 + 3$

(b) $\frac{7}{1} + \frac{3}{1}$

(c) $\frac{14}{2} + \frac{2}{3}$

(d) $0 + 7$

(e) $\frac{0}{1} + \frac{7}{1}$

(f) $-15 + 7$

(g) $\frac{-15}{1} + \frac{-7}{1}$

(h) $-8 + (-4)$

(i) $\frac{-8}{1} + \frac{-4}{1}$

9. Can you describe an isomorphism which the sums in Exercise 8 suggest?

10. (a) Is (Z,+) a group? If so, is it commutative?
(b) Is (Q,+) a group? If so, is it commutative?

11. Give the additive inverse of each of the following rational numbers.

(a) $\frac{2}{3}$

(d) $\frac{15}{7}$

(b) $\frac{-5}{3}$ (e) $\frac{-15}{7}$

(c) $\frac{0}{1}$ (f) $\frac{-a}{b}$

12. If we use " $-\frac{a}{b}$ " to denote the additive inverse of the rational number $\frac{a}{b}$, complete each of the following so as to have a true statement.

(a) $-\frac{3}{4} =$ (d) $-\frac{-75}{7} =$ (g) $-\frac{-7}{8} =$

(b) $-\frac{-5}{2} =$ (e) $-\frac{2}{5} =$ (h) $-(-\frac{-7}{8}) =$

(c) $-\frac{10}{3} =$ (f) $-(-\frac{2}{5}) =$ (i) $-(-\frac{a}{b}) =$

13. Compute the following.

(a) $\frac{1}{2}(\frac{1}{6} + \frac{7}{6})$

(b) $(\frac{1}{2} \cdot \frac{1}{6}) + (\frac{1}{2} \cdot \frac{7}{6})$

(c) $\frac{2}{3}(\frac{5}{8} + \frac{3}{8})$

(d) $(\frac{2}{3} \cdot \frac{5}{8}) + (\frac{2}{3} \cdot \frac{3}{8})$

(e) $(\frac{7}{5} + \frac{-3}{5})\frac{3}{4}$

(f) $(\frac{7}{5} \cdot \frac{3}{4}) + (\frac{-3}{5} \cdot \frac{3}{4})$

(g) $\frac{2}{5}(\frac{1}{2} + \frac{2}{3})$

(h) $(\frac{2}{5} \cdot \frac{1}{2}) + (\frac{2}{5} \cdot \frac{2}{3})$

14. On the basis of the computations in Exercise 13, how do you think the following should be completed:

$\frac{a}{b}(\frac{c}{d} + \frac{e}{f}) =$

What property is this a statement of?

*15. If $\frac{a}{b}$, $\frac{c}{d}$, and $\frac{e}{f}$ are three rational numbers, can you give an argument showing that the distributive property holds? (Do not use specific numbers.)

12.15 Subtraction of Rational Numbers

In (Z, +), we say, for example

$5 - 3 = 2$, because $2 + 3 = 5$.

And, in general,

if $c + b = a$, then $a - b = c$.

In other words, subtraction is defined in terms of addition. We shall make the same sort of definition in (Q, +). For instance,

since $\frac{2}{5} + \frac{1}{5} = \frac{3}{5}$, we agree that

$$\frac{3}{5} - \frac{1}{5} = \frac{2}{5}.$$

And $\frac{2}{5}$ is the *difference* between $\frac{3}{5}$ and $\frac{1}{5}$, or the result of *subtracting* $\frac{1}{5}$ from $\frac{3}{5}$. We could have found this difference in the following way:

$$\frac{2}{5} + \frac{-1}{5} = \frac{1}{5}.$$

That is, instead of subtracting $\frac{1}{5}$, we might add the additive inverse of $\frac{1}{5}$; this is of course the same pattern we noticed earlier for the integers.

We consider below the general case for the rational numbers.

$$\text{Let } \frac{a}{b} - \frac{c}{d} = \frac{x}{y}.$$

Then by definition of subtraction,

$$\frac{x}{y} + \frac{c}{d} = \frac{a}{b}$$

$$(\frac{x}{y} + \frac{c}{d}) + \frac{-c}{d} = \frac{a}{b} + \frac{-c}{d}$$

$$\frac{x}{y} + (\frac{c}{d} + \frac{-c}{d}) = \frac{a}{b} + \frac{-c}{d}$$

$$\frac{x}{y} + \frac{0}{1} = \frac{a}{b} + \frac{-c}{d}$$

$$\frac{x}{y} = \frac{a}{b} + \frac{-c}{d}$$

But in our original equation,

$$\frac{x}{y} = \frac{a}{b} - \frac{c}{d}.$$

Therefore,

$$\frac{a}{b} - \frac{c}{d} = \frac{a}{b} + \frac{-c}{d}.$$

As a practical matter then we can always find a sum instead of a difference, provided we remember to add the inverse of the number being subtracted.

$$\begin{aligned} \text{Example: } \frac{-3}{5} - \frac{-2}{3} &= \frac{-3}{5} + \frac{2}{3} \\ &= \frac{-9}{15} + \frac{10}{15} \\ &= \frac{-9 + 10}{15} \\ &= \frac{1}{15} \end{aligned}$$

12.16 Exercises.

1. Compute the following differences.

(a) $\frac{3}{5} - \frac{1}{5}$

(b) $\frac{10}{13} - \frac{5}{13}$

(c) $\frac{5}{13} - \frac{10}{13}$

(d) $\frac{2}{3} - \frac{11}{3}$

(e) $\frac{3}{5} - \frac{-1}{5}$

(f) $\frac{-3}{5} - \frac{1}{5}$

(g) $\frac{-3}{5} - \frac{-1}{5}$

(h) $\frac{7}{8} - \frac{3}{4}$

(i) $\frac{6}{8} - \frac{3}{4}$

(j) $\frac{5}{6} - \frac{21}{8}$

(k) $\frac{5}{6} - \frac{-21}{8}$

(l) $\frac{-5}{6} - \frac{21}{8}$

(m) $\frac{-5}{6} - \frac{-21}{8}$

(n) $\frac{2}{13} - \frac{3}{15}$

2. (a) What is the difference $\frac{2}{3} - \frac{3}{5}$?

(b) What is the sum $\frac{2}{3} + \frac{-3}{5}$?

(c) What number does " $-\frac{3}{5}$" name?

(d) What is the sum $\frac{2}{5} + (-\frac{3}{5})$?

3. Compute the following:

(a) $\frac{7}{8} - \frac{3}{8}$

(b) $\frac{7}{8} + \frac{-3}{8}$

(c) $\frac{7}{8} + (-\frac{3}{8})$

(d) $\frac{5}{12} + (-\frac{13}{16})$

(e) $\frac{5}{3} + (-\frac{3}{7})$

(f) $\frac{5}{3} - \frac{3}{7}$

(g) $\frac{7}{12} + (-\frac{7}{12})$

(h) $\frac{7}{12} - \frac{7}{12}$

4. Is subtraction a binary operation on the set Q of rational numbers?

5. (a) Is subtraction of rational numbers associative?
   (b) Is subtraction of rational numbers commutative?
   (c) Is there an identity element in (Q,−)?

6. Is (Q,−) a group? Why or why not?

12.17 Ordering the Rational Numbers.

In the set Z of integers, we know that

$$2 < 3.$$

Therefore, in ordering the rational numbers, we would like to be able to make statements such as the following:

$$\frac{2}{1} < \frac{3}{1}$$

$$\frac{4}{2} < \frac{6}{2}$$

$$\frac{6}{3} < \frac{9}{3}$$

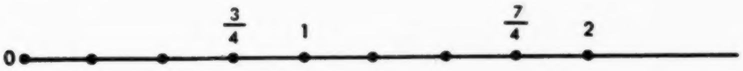
since we have already noticed that there is a close relationship between such integers as 2 and 3, and such rational numbers such as $\frac{2}{1}$ and $\frac{3}{1}$. If this relationship is to hold for ordering also, the examples above suggest that we agree to the following:

In Q, $\frac{a}{b} < \frac{c}{b}$ if and only if $a < c$ in Z

(We are assuming that b is a positive integer.)

Example 1. $\frac{3}{4} < \frac{7}{4}$ in Q, since $3 < 7$ in Z.

Notice that if we represent the rational numbers $\frac{3}{4}$ and $\frac{7}{4}$ on a number scale, the point representing $\frac{3}{4}$ is to the left of the point representing $\frac{7}{4}$.



Example 2. Compare the rational numbers $\frac{11}{13}$ and $\frac{7}{9}$.

Which is less? Our method for comparing rational numbers is based on fractions that have the same denominator. Therefore, we shall use the fractions

$$\frac{11 \cdot 9}{13 \cdot 9} \text{ and } \frac{7 \cdot 13}{9 \cdot 13}$$

to compare the given rational numbers. (Do you see why these fractions were chosen?)

Now, since $7 \cdot 13 < 11 \cdot 9$ in Z, we have

$$\frac{7}{9} < \frac{11}{13} \text{ in Q.}$$

From Example 2, we notice that $\frac{7}{9} < \frac{11}{13}$ since $7 \cdot 13 < 9 \cdot 11$. And this suggests a general way of comparing two rational numbers without actually writing fractions with the same denominator. Suppose $\frac{a}{b}$ and $\frac{c}{d}$ are two rational numbers (and b and d are both positive integers). Then the fractions $\frac{ad}{bd}$ and $\frac{bc}{bd}$ also represent these numbers. (Why?) And by our earlier agreement,

$$\frac{ad}{bd} < \frac{bc}{bd} \text{ if and only if } ad < bc.$$

Therefore, we make the following definition for ordering rational numbers:

If $\frac{a}{b}$ and $\frac{c}{d}$ are rational numbers,
and b and d are positive integers
$\frac{a}{b} < \frac{c}{d}$ if and only if $ad < bc$.

Example 3. Compare the rational numbers $\frac{2}{3}$ and $\frac{4}{5}$.

Since $2 \cdot 5 < 3 \cdot 4$, $\quad \frac{2}{3} < \frac{4}{5}$.

In the definition above and in all of our examples, ve have demanded that the denominators of the fractions sed in comparing rational numbers be positive. Is this ecessary?

Consider the rational numbers $\frac{2}{1}$ and $\frac{3}{1}$.

Ve have already agreed that $\frac{2}{1} < \frac{3}{1}$, since $2 < 3$.

And yet if we were to use the fractions $\frac{-2}{-1}$ and $\frac{-3}{-1}$ o represent these numbers, it is *not true* that $-2 < -3$. This illustrates the importance of using ractions with *positive denominators* when comparing ational numbers:

Questions: Can every rational number be represented by a fraction with a positive denominator?

What fraction with positive denominator represents the same rational number as $\frac{3}{-2}$? as $\frac{-7}{-3}$?

2.18 Exercises.

1. Represent the rational numbers in each pair below by fractions have the same denominator. Then decide which rational number is less.

(a) $\frac{2}{5}$ and $\frac{3}{8}$ (c) $\frac{5}{4}$ and $\frac{7}{5}$
(b) $\frac{3}{4}$ and $\frac{5}{8}$ (d) $\frac{8}{3}$ and $\frac{9}{4}$

2. Draw a number scale, and locate a point to represent each of the rational numbers in Exercise 1.

3. Decide which of the following statements are true, and which are false. (as with the integers, the sign " > " means "is greater than.")

(a) $\frac{7}{5} < \frac{3}{2}$ (d) $\frac{-2}{7} < \frac{-3}{8}$ (g) $\frac{-3}{2} > \frac{-7}{4}$
(b) $\frac{1}{2} > \frac{5}{7}$ (e) $\frac{0}{1} > \frac{5}{7}$ (h) $\frac{2}{8} > \frac{1}{4}$
(c) $\frac{23}{15} < \frac{15}{10}$ (f) $\frac{-2}{7} < \frac{0}{1}$ (i) $\frac{85}{32} < \frac{31}{12}$

4. For each pair of rational numbers below, decide which is less.

(a) $\frac{1}{2}; \frac{5}{8}$ (d) $-\frac{11}{23}; \frac{-7}{15}$ (g) $\frac{100}{51}; \frac{13}{7}$
(b) $-\frac{1}{2}; \frac{-5}{8}$ (e) $\frac{8}{-3}; \frac{14}{-5}$ (h) $\frac{0}{1}; \frac{0}{52}$
(c) $\frac{11}{23}; \frac{7}{15}$ (f) $\frac{12}{9}; \frac{4}{3}$ (i) $\frac{5}{-3}; \frac{-7}{4}$

5. If $\frac{a}{b} > \frac{0}{1}$, then $\frac{a}{b}$ is a *positive* rational number.

If $\frac{a}{b} < \frac{0}{1}$, then $\frac{a}{b}$ is a *negative* rational number.

Decide whether each of the following rational numbers is positive, negative, or zero.

(a) $\frac{5}{2}$ (e) $\frac{12}{-7}$ (i) $\frac{0}{8}$
(b) $\frac{-5}{2}$ (f) $\frac{-3}{-4}$ (j) $\frac{10}{-3}$
(c) $\frac{5}{-2}$ (g) $\frac{-11}{5}$ (k) $\frac{-3}{-10}$
(d) $\frac{-5}{-2}$ (h) $\frac{4}{3}$ (i) $\frac{1211}{315}$

6. If $\frac{a}{b}$ is a rational number, and the product of the integers *a* and *b* is a positive integer, is the rational number $\frac{a}{b}$ positive? Give an argument for your answer.

7. Answer each of the following, and give an argument for your answer.

(a) Does the ordering of the rational numbers possess the *reflexive* property?

(b) Does the ordering of the rational numbers possess the *symmetric* property?

(c) Does the ordering of the rational numbers possess the *transitive* property?

8. Complete the following sentences:

(a) If $\frac{a}{b} < \frac{c}{d}$, then ad__bc.

(b) If $\frac{a}{b} = \frac{c}{d}$, then ad__bc.

(c) If $\frac{a}{b} > \frac{c}{d}$, then ad__bc.

9. (a) Is there an *integer* " between" 2 and 3? That is, is there an integer x such that $2 < x$ and $x < 3$? If so, name one.

(b) Is there a rational number between $\frac{2}{1}$ and $\frac{3}{1}$? If so, name one.

(c) Name a rational number between $\frac{2}{3}$ and $\frac{3}{4}$. (Hint: You might find the "average" of the numbers.)

(d) Name a rational number between $\frac{5}{4}$ and $\frac{7}{5}$.

(e) Given any two rational numbers, do you think it is possible to find another rational number that is between them? Give an argument for your answer.

10. If $\frac{a}{b} < \frac{c}{d}$ and $\frac{c}{d} < \frac{a}{b}$, what conclusion can you make about $\frac{a}{b}$ and $\frac{c}{d}$?

12.19 Integers and Rational Numbers: An Isomorphism.

Throughout this chapter, we have commented on the close relationship between the integers and certain rational numbers. To illustrate what we mean by this, look at the statements below. The ones on the left are about *integers:* the ones on the right are about *rational numbers*.

In $(Z, +)$, $3 + 2 = 5$ In $(Q, +)$, $\frac{3}{1} + \frac{2}{1} = \frac{5}{1}$

In $(Z, \cdot)$, $3 \cdot 2 = 6$ In $(Q, \cdot)$, $\frac{3}{1} \cdot \frac{2}{1} = \frac{6}{1}$

In Z, $2 < 3$ In Q, $\frac{2}{1} < \frac{3}{1}$

Now the similarities between these statements do not occur because we used the particular integers 2 and 3. We could in fact let a and b represent any two integers at all. Corresponding to them are the rational numbers $\frac{a}{1}$ and $\frac{b}{1}$; and we have the following statements:

If $a + b = c$ in $(Z, +)$, then $\frac{a}{1} + \frac{b}{1} = \frac{c}{1}$ in $(Q, +)$.

If $a \cdot b = d$ in $(Z, \cdot)$, then $\frac{a}{1} \cdot \frac{b}{1} = \frac{d}{1}$ in $(Q, \cdot)$.

If $a < b$ in Z, then $\frac{a}{1} < \frac{b}{1}$ in Q.

Each of these statements can be proved by the way we have defined addition, multiplication, and ordering of rational numbers; but we shall not give the proof here. By this time you may recognize a kind of pattern we saw earlier with the whole numbers and certain integers. That is, in the set Q there is a "copy" of the integers. There is a set of rational numbers whose behavior copies so closely the behavior of the integers that we can use integer names for them without causing confusion.

For example, we may write "$2 \cdot 3 = 6$" instead of "$\frac{2}{1} \cdot \frac{3}{1} = \frac{6}{1}$". And we can write "$5 \cdot \frac{2}{3} = \frac{10}{3}$" instead of "$\frac{5}{1} \cdot \frac{2}{3} = \frac{10}{3}$".

In other words, to use language that we used earlier, we can say that the integers are *isomorphic* to the set of rational numbers that are of the form $\frac{a}{1}$.

12.20 Exercises.

In problems 1 – 20, make the indicated rational number computations.

1. $3 + \frac{7}{8} =$
2. $7 \cdot \frac{2}{7} =$
3. $7 \cdot \frac{7}{2} =$
4. $\frac{3}{8} - 2 =$
5. $\frac{-3}{8} - 2 =$
6. $\frac{-3}{8} \div 2 =$
7. $2 \div \frac{-3}{8}$
8. $(\frac{3}{4} \cdot 9) \cdot \frac{6}{7}$
9. $2 \div (\frac{1}{2} \div 3)$
10. $(2 \div \frac{1}{2}) \div 3$
11. $-2 \cdot (\frac{3}{8} \cdot 8)$
12. $(-2 \cdot \frac{3}{8}) \cdot 8$
13. $21 - \frac{3}{4}$
14. $\frac{3}{4} - 21$
15. $\frac{-2}{3} \cdot 3$
16. $\frac{2}{3} \cdot -3$
17. $\frac{5}{4} \cdot 3 \cdot \frac{2}{15} \cdot 10$
18. $(\frac{2}{3} \cdot 6) + 3 \quad \div \frac{1}{2}$
19. $(2 \div \frac{1}{3}) + (\frac{1}{3} \div 2)$
20. $(5 \cdot \frac{3}{5})^2 \quad + (\frac{-2}{3} \cdot 3)$

In each of the problems 21 – 26, decide which of the rational numbers in the pair is less.

21. $-3; \frac{-7}{3}$
22. $14; \frac{41}{3}$
23. $\frac{21}{5}; 4$
24. $\frac{-21}{5}; -4$
25. $6; \frac{47}{8}$
26. $1; \frac{999}{1000}$

12.21 Decimal Fractions.

In the preceding sections, we have developed the system $(Q, +, \cdot)$. Now we look at another way of naming rational numbers, a way that is based on the idea of *place value*. You are probably already familiar with the idea of place value; for instance, when we write "3507," we mean

$(3 \cdot 1000) + (6.100) + (0 \cdot 10) + (7 \cdot 1)$, or

$(3 \cdot 10^3) + (6 \cdot 10^2) + (0 \cdot 10^1) + (7 \cdot 1)$.

This form is sometimes referred to as "expanded notation."

In fact, from your work in elementary school, you have probably seen charts as the one below which explain the place value scheme used in writing names of rational numbers that are also whole numbers.

1	5	4	8	7	6	3
10^6	10^5	10^4	10^3	10^2	10	1
MILLIONS	HUNDRED THOUSANDS	TEN THOUSANDS	THOUSANDS	HUNDREDS	TENS	ONES

hus, in "1,548,763," the "7" represents 7 *hundreds* hat is, 700), since it is in the "third place" to the *eft* of the decimal point. (In writing the name of a vhole number, it is not common to mark the decimal oint, but it is at the extreme right.) There is a very mportant pattern in this place value scheme. As you nove from *left to right*, the value associated with each lace is $\frac{1}{10}$ of the value associated with the preceding lace. Thus, with the third place we associate the alue 100; but with the second place, we associate he value $\frac{1}{10} \cdot 100$, or 10. In order to have names for all ational numbers (not just whole numbers) we extend his pattern to the *right* of the decimal point. That is, he value of the first place to the *right* of the decimal oint is $\frac{1}{10} \cdot 1$, or $\frac{1}{10}$; the value of the second place o the right of the decimal point is $\frac{1}{10} \cdot \frac{1}{10}$, or $\frac{1}{100}$. We may also indicate $\frac{1}{100}$ as $\frac{1}{10^2}$. The table below shows the values associated with the first six places to the right of the decimal point. (You should be able to extend the table as far to the right as desired.)

3	4	0	7			
$\frac{1}{10}$	$\frac{1}{10^2}$	$\frac{1}{10^3}$	$\frac{1}{10^4}$	$\frac{1}{10^5}$	$\frac{1}{10^6}$	
TENTHS	HUNDREDTHS	THOUSANDTHS	TEN THOUSANDTHS	HUNDRED THOUSANDTHS	MILLIONTHS	

In the table you see the numeral ".3407," and the table makes it easy to see that this means

$$(3 \cdot \frac{1}{10}) + (4 \cdot \frac{1}{100}) + (0 \cdot \frac{1}{1000}) + (7 \cdot \frac{1}{10000}).$$

But this is also

$$\frac{3000}{10000} + \frac{400}{10000} + \frac{0}{10000} + \frac{7}{10000} = \frac{3407}{10000}$$

(Do you see why?)

Therefore,

$$.3407 = \frac{3407}{10000};$$

and ".3407" is a *decimal fraction* name for a rational number.

Question: Can you write an equation of form "$b \cdot x = a$" whose solution is the rational number .3407?

If you are not already familiar with decimal fraction notation, the following examples should help to make it clear.

Example 1. ".25" is the name of a rational number. Represent this rational number by an irreducible fraction.

We know that $.25 = (2 \cdot \frac{1}{10}) + (5 \cdot \frac{1}{100})$

$$= \frac{20}{100} + \frac{5}{100}$$

$$= \frac{25}{100}.$$

Of course, $\frac{25}{100}$ is not an irreducible fraction. But we know that $\frac{25}{100} = \frac{1}{4}$. Therefore,

$$.25 = \frac{1}{4}.$$

Example 2. Represent the rational number .250 by an irreducible fraction.

$$.250 = \frac{250}{1000} = \frac{25}{100}.$$

Do you see then that this example is really the same as Example 1? Again, the irreducible fraction called for is $\frac{1}{4}$. That is,

$$.250 = .25 = \frac{1}{4}.$$

On the basis of Example 2, you should begin to see why it is true that some rational numbers have an infinite number of decimal fraction representations. Thus, $\frac{1}{4} = .25 = .250 = .2500 = .25000$, etc.

Question: Do the decimal fractions .4 and .400 represent the same number? Why or why not?

Example 3. Represent the number 4.18 by a fraction $\frac{a}{b}$, where a and b are integers.

$4.18 = 4 + \frac{18}{100}$. But $4 = \frac{400}{100}$.

So, $4.18 = \frac{400}{100} + \frac{18}{100}$

$$= \frac{418}{100}.$$

Example 4. Represent the rational number $\frac{2}{5}$ by a decimal fraction.

We know $\frac{2}{5} = \frac{4}{10}$. (Why?) Therefore, $\frac{2}{5} = .4$

Of course, we could also use ".40," ".400," ".4000," etc.

Example 5. Represent $15\frac{2}{5}$ by a decimal fraction.

An expression such as "$15\frac{2}{5}$" is sometimes called a mixed *numeral*, since it looks as though it is composed of a symbol for a whole number together with a fraction. The important point to understand is that it means

$$15 + \frac{2}{5}.$$

Therefore, from Example 4, we know

$$\begin{aligned} 15\tfrac{2}{5} &= 15 + .4 \\ &= 15.4 \end{aligned}$$

Example 6. Represent $\frac{3}{8}$ by a decimal fraction.

We know that $\frac{3}{8}$ is a quotient; namely, $3 \div 8$.

Therefore, in the space at the right, we carry out this division. Another way to think about this division is as follows:

$$\begin{array}{r} .375 \\ 8\,\overline{)\,3.000} \\ \underline{2\,4} \\ 60 \\ \underline{56} \\ 40 \\ 40 \end{array}$$

$$\begin{aligned} 1000 \cdot \frac{3}{8} &= \frac{3000}{8} \\ &= 375. \end{aligned}$$

Then, since $1000 \cdot \frac{3}{8} = 375$, $\frac{3}{8} = \frac{375}{1000}$. (Do you remember how a rational number was defined as the solution of an equation?)

12.22 Exercises

1. Express each of the following decimal fractions as an irreducible fraction $\frac{a}{b}$.

(a) .3
(b) .32
(c) .320
(d) .325
(e) 7.3
(f) .03
(g) .003
(h) .000003
(i) .500
(j) .005
(k) 3.05
(l) 25.1
(m) .625
(n) 10.625
(o) .33

2. We know that every rational number is the solution of an equation of the form "$b \cdot x = a$," where a and b are integers, $b = 0$. For each of the following rational numbers, write an equation of which the number is the solution.

Example: $.19 = \frac{19}{100}$

Therefore, .19 is the solution of

"$100 \cdot x = 19$."

(a) .5
(b) .7
(c) .08
(d) .07
(e) .33
(f) .333
(g) 2.7
(h) .375
(i) .60
(j) .6
(k) .123456
(l) .333333
(m) −.5
(n) −.05
(o) −2.7
(p) −.375

3. Find a decimal fraction name for each of the following rational numbers. (The rational numbers listed in this exercise are so frequently used that it is advisable to remember their decimal fraction representations.)

(a) $\frac{1}{2}$
(b) $\frac{1}{4}$
(c) $\frac{3}{4}$
(d) $\frac{1}{5}$
(e) $\frac{2}{5}$
(f) $\frac{3}{5}$
(g) $\frac{4}{5}$
(h) $\frac{1}{8}$
(i) $\frac{3}{8}$
(j) $\frac{5}{8}$
(k) $\frac{7}{8}$

4. For each of the following decimal fractions, write *four other decimal fractions* which represent the same number.

(a) .5
(b) .3
(c) .05
(d) 25.6
(e) 4.0
(f) .025
(g) .000005
(h) .25
(i) 5

5. Recall that a rational number is one which can be represented as a quotient $\frac{a}{b}$, where a and b, the numerator and denominator, *are integers.*

(a) In the decimal fraction ".5," what is the numerator? What is the denominator?

(b) What are the numerator and denominator of ".00007"?

(c) What are the numerator and denominator of "8.2"?

(d) Does every decimal fraction represent a rational number? Explain. (How is the numerator determined? How is the denominator determined?)

6. Find a decimal fraction which represents each of the following rational numbers. (See Example 6 in the text.)

(a) $\frac{3}{20}$
(b) $\frac{7}{20}$
(c) $\frac{8}{25}$
(d) $\frac{21}{25}$
(e) $\frac{25}{64}$
(f) $\frac{63}{200}$

12.23 Infinite Repeating Decimals.

Can every rational number be represented by a *decimal fraction*? The exercises in the preceding section may lead you to answer "yes", and although this is correct, there is a major difficulty with many rational numbers. As an example, let us try to find a decimal fraction for $\frac{1}{3}$. As before, we know this is a quotient, and the appropriate division is shown below:

$$\begin{array}{r l} & .3333\ldots\ldots \\ 3 & \overline{)\,1.0000} \\ & \underline{9} \\ & 10 \\ & \underline{9} \\ & 10 \\ & \underline{9} \\ & 10 \\ & \underline{9} \\ & 1 \end{array}$$

Do you see the difficulty? In this case, the division process is something like a broken record. For, as long as we care to continue writing, we will have to place a '3' in each place to the right of the decimal point. Thus, this decimal does not "end" or "terminate" as it does, for example with $\frac{3}{8} = .375$. (See Example 6 of Section 12.21)

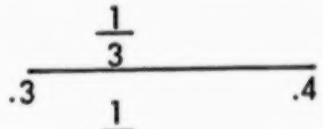
How then can we represent $\frac{1}{3}$ with a decimal fraction? One answer lies in giving an *approximate* decimal fraction. To see this, study the following steps.

$$0 < \frac{1}{3} < 1$$

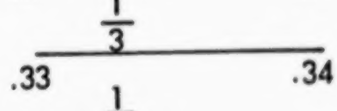
We know that $\frac{1}{3}$ is "between" 0 and 1, and we say that $\frac{1}{3}$ is in the *closed interval* $[0,1]$. In terms of a number scale, this means that the point representing $\frac{1}{3}$ lies on that part of the line consisting of the points representing 0 and 1, together with all the points between those two:

We can also place $\frac{1}{3}$ in smaller and smaller intervals, as follows:

$$.3 < \frac{1}{3} < .4$$



$$.33 < \frac{1}{3} < .34$$



$$.333 < \frac{1}{3} < .334$$

$$.3333 < \frac{1}{3} < .3334$$

1/3
.3333
.3334

Do you see that in a way we are "squeezing" the number $\frac{1}{3}$? Each of the above intervals is "smaller" than the one before it, and is contained in it. We call such intervals *nested intervals*. Thus, we have a sequence of nested intervals containing the rational number $\frac{1}{3}$. Although we stopped with the interval $[.3333, .3334]$, the sequence goes on without end.

Question: Continuing in the pattern above, what is the "next" interval in this sequence of nested intervals?

If we form a sequence of the first numbers in these nested intervals, we get: .3, .33, .333, .3333, .33333, . . . , a sequence of rational numbers. None of the numbers in this sequence is equal to $\frac{1}{3}$. For instance, consider the first number, .3:

$.3 \neq \frac{1}{3}$. In fact, $.3 < \frac{1}{3}$. We can find the *difference* between $\frac{1}{3}$ and .3 as follows:

$$\frac{1}{3} - .3 = \frac{1}{3} - \frac{3}{10}$$
$$= \frac{10}{30} - \frac{9}{30} = \frac{1}{30}.$$

Therefore, although $.3 \neq \frac{1}{3}$, it is "very close" to $\frac{1}{3}$, because the difference between the numbers is small.'''' We can say that .3 is an approximation to $\frac{1}{3}$, and write

$$\frac{1}{3} \approx .3$$

This approximation is said to be correct to *tenths* or "to one decimal place."

Next let us consider the second number in the sequence, .33. The difference between this number and $\frac{1}{3}$ is computed below:

$$\frac{1}{3} - .33 = \frac{1}{3} - \frac{33}{100}$$
$$= \frac{100}{300} - \frac{99}{300}$$
$$= \frac{1}{300}.$$

Therefore, .33 is a "better approximation" to $\frac{1}{3}$ than is .3. That is, it is "closer" to $\frac{1}{3}$ since it differs from it by only $\frac{1}{300}$ instead of $\frac{1}{30}$. (How do we know that $\frac{1}{300} < \frac{1}{30}$?). Thus we write

$$\frac{1}{3} \approx .33,$$

and say that this approximation is correct to *hundredths* or "to two decimal places."

In fact, as you might have guessed, each number in the sequence above is a closer approximation to $\frac{1}{3}$ that the number preceding it.

Question: What is the difference between $\frac{1}{3}$ and .333 ?

And though we shall not explore the matter here, it is true that by "going far enough in the sequence" you can get a number as close to $\frac{1}{3}$ as you like.

Now, from the number $\frac{1}{3}$, we have learned a very important fact. Not every rational number can be expressed by a *terminating* decimal fraction. Many rational numbers, such as $\frac{1}{3}$, have decimal fraction representations that are *infinite, repeating* decimals. They might be called "rubber stamp" decimals also; for example, if you had a rubber stamp made with the digit '3' on it, you could write the decimal fraction for $\frac{1}{3}$ by just stamping the "3" over and over again.

As another example, let us work with the rational number $\frac{8}{33}$.

$$\begin{array}{r}
.2424\ldots \\
33\,\overline{)\,8.0000} \\
\underline{6\,6} \\
1\,40 \\
\underline{1\,32} \\
80 \\
\underline{66} \\
140 \\
\underline{132} \\
8
\end{array}$$

$.24 < \frac{8}{33} < .25 \qquad \frac{8}{33} \approx .24$ (correct to hundredths)

$.2424 < \frac{8}{33} < .2425 \qquad \frac{8}{33} \approx .2424$ (to four decimal places)

$.242424 < \frac{8}{33} < .242425 \quad \frac{8}{33} \approx .242424$

As with $\frac{1}{3}$, there is no terminating decimal representation for $\frac{8}{33}$, but there is an infinite repeating decimal associated with it; and we can approximate $\frac{8}{33}$ to any desired number of decimal places.

12.24 Exercises.

1. (a) What is the difference between $\frac{1}{3}$ and .333?

   (b) What is the difference between $\frac{1}{3}$ and .3333?

   (c) Which of the numbers, .333 and .3333, is a better approximation to $\frac{1}{3}$?

2. (a) Write an equation of the form "$b \cdot x = a$" which has $\frac{1}{3}$ as solution.

   (b) Write an equation of the form "$b \cdot x = a$" which has .3 as solution.

   (c) Write an equation of the form "$b \cdot x = a$" which has .33 as solution.

   (d) Would the same equation work for all of the parts (a), (b), and (c)? Why or why not?

3. In looking for a decimal fraction representation of $\frac{1}{6}$, the division process below might be used:

$$\begin{array}{r}
.1666\ldots \\
6\,\overline{)\,1.0000} \\
\underline{6} \\
40 \\
\underline{36} \\
40 \\
\underline{36} \\
40 \\
\underline{36} \\
4
\end{array}$$

Thus, we again get an infinite repeating decimal, although the digits do not start repeating right away.

Now answer the following questions:

(a) What is the difference between $\frac{1}{6}$ and .16?

(b) What is the difference between $\frac{1}{6}$ and .17?

(c) Which is a better approximation to $\frac{1}{6}$, .16 or .17?

(d) What is the difference between $\frac{1}{6}$ and .166?

(e) What is the difference between $\frac{1}{6}$ and .167?

(f) Which is a better approximation to $\frac{1}{6}$, .166 or .167?

(g) Which is a better approximation to $\frac{1}{6}$, .17 to .167?

(h) What is the best approximation to $\frac{1}{6}$, correct to *four decimal places*?

4. For each of the following rational numbers, write

the best approximation decimal fraction approximation, correct to four decimal places.

(a) $\frac{5}{6}$ (c) $\frac{1}{11}$ (e) $\frac{1}{12}$

(b) $\frac{2}{3}$ (d) $\frac{2}{11}$ (f) $\frac{5}{12}$

5. Consider the sequence below:

   .1, .11, .111, .1111, ...

   (a) What is the difference between $\frac{1}{9}$ and .1?

   (b) What is the difference between $\frac{1}{9}$ and .11?

   (c) What is the difference between $\frac{1}{9}$ and .111?

   (d) What is the difference between $\frac{1}{9}$ and .1111?

   (e) Suppose the sequence continues in the same pattern suggested by the first four terms. How far would you have to go in the sequence to find a number that differs from $\frac{1}{9}$ by $\frac{1}{9{,}000{,}000}$?

6. (a) Give an approximate decimal fraction (correct to three decimal places) for the rational number $2\frac{1}{3} = \frac{7}{3}$.

   (b) Is the decimal fraction representation of $2\frac{1}{3}$ an infinite repeating decimal?(Remember that the decimal fraction need not start repeating right away.)

7. Consider the quotient $\frac{1}{7}$.

   (a) In dividing by 7, how many numbers are *possible* as remainders? (Remember that a remainder must be less than the divisor.)

   (b) Carry out the division process for $1 \div 7$ to twelve decimal places.

   (c) At what stage in the division process did you get a remainder that had occurred before?

   (d) At what stage in the division process did the decimal fraction start "repeating"? Can you explain why it happened at that particular time?

8. In carrying out the division $3 \div 8$, what remainder occurs that causes the decimal fraction to *terminate?*

9. Try to give a convincing argument for the following:

   The decimal fraction representation for any rational number $\frac{a}{b}$ is either a terminating decimal or an infinite repeating decimal.

10. Write a sequence of *nested intervals* all of which contain the number $\frac{1}{11}$. Begin with the interval [0,1] and get a total of five intervals. Also show the intervals on a number scale.

11. Explain why the following sequence of intervals is *not* a nested sequence:

    [0,1], [1,2] [1,5; 2.5], [.1, .2]

12.25 Decimal Fractions and Order of the Rational Numbers.

We have already seen how to tell which of two rational numbers $\frac{a}{b}$ and $\frac{c}{d}$ is less, when fractions are used to represent the numbers. Now let us see how to make such a comparison when decimal fractions are used.

Example 1. Which is less, .3 or .4?

Since $.3 = \frac{3}{10}$, and $.4 = \frac{4}{10}$,

it is easy to tell that $.3 < .4$.

Example 2. Which is less, .2567 or .2563? Notice that the first three digits of these decimal fractions agree, place by place. The fourth decimal place is the first one in which they differ.

$$.2567 = \frac{256}{1000} + \frac{7}{10000};$$

$$.2563 = \frac{256}{1000} + \frac{3}{10000}.$$

Therefore, $.2563 < .2567$.

Example 3. Which is less, .8299 or .8521?

$$.8299 = \frac{8}{10} + \frac{299}{10000}$$

$$.8521 = \frac{8}{10} + \frac{521}{10000}$$

Therefore, $.8299 < .8521$. Notice again that these two decimal fractions agree in the first decimal place. The first place in which they disagree is the second place; and $2 < 5$.

These three examples show that it is very easy to tell which of two rational numbers is less when the numbers are represented by decimal fractions. Suppose we have two decimal fractions

$$.a_1a_2a_3a_4$$

and

$$. \, . \, b_1 b_2 b_3 b_4$$

and $a_1 = b_1$, $a_2 = b_2$, but $b_3 < a_3$. Then do you see that $.b_1b_2b_3b_4 < .a_1a_2a_3a_4$? In other words, the way to tell which of two decimal fractions represents the smaller number is to look for the first place (reading from left to right) in which they disagree; the one which has the smaller digit in that place represents the smaller number.

Example 4. Which is less, 23.524683 or 23.524597? The first place in which these decimal fractions "disagree" is the fourth decimal place. And since $5 < 6$, then $23.524597 < 23.524683$.

12.26 Exercises.

1. In each of the following, write the two decimal fractions. Then place either a "<" or a ">" or a "=" between them so that a true statement results.

(a) 12.5 12.4
(b) 8.33 8.34
(c) .1257 .1250
(d) .1257 .125
(e) .6666 .6667
(f) 826.33 826.30
(g) 5.4793293 5.4789999
(h) 548 551
(i) 1.9999 2
(j) .9874 .9875

2. This exercise is similar to exercise 1, except that negative rational numbers are used. Remember, that although $1 < 2$, for instance, $-2 < -1$. Thus, although $.5 < .6$, we have $-.6 < -.5$.

(a) −3.567 −3.582
(b) −.12345 −12453
(c) −.99 −1
(d) −100.555 − 100.565
(e) −42.80 −42.85
(f) −42.8 −42.85
(g) −12.9999 12.9998
(h) −4.378 −4.3779

3. Is it possible to find a rational number x "between" .354 and .357? That is, we want a number x such that

$$.354 < x < .357.$$

Notice that these two decimal fractions agree in the first two places, but disagree in the third place. Thus, for x, we can use a decimal fraction that agrees with the two given ones in the first two places, but has in the third place a digit that is between the two given third digits. For example, x might be .355, since $.354 < .355 < .357$. (This is not the only value of x that can be used. Can you give others?)

Now for each pair of rational numbers below, name a rational number that is between them.

(a) .6; .8
(b) 2.35; 2.39
(c) 45.987; 45.936
(d) 102; 108
(e) 5.420 5.430
(f) 5.42 5.43
(g) 3.8 3.9
(h) 2.99 3

Compare Exercise 3 with Exercise 9 in Section 12.18. Do you see that between the two rational numbers it is always possible to find another rational number? For this reason, we say that (Q, <) is *dense:* that is, the rational numbers form a *dense set.*

4. Given the rational numbers 1 and 2, find a rational number x such that $1 < x < 2$; then find a rational number y such that $1 < y < x$; then find a rational number z such that $1 < z < y$; then find a rational number w such that $1 < w < z$; Draw a number scale, and represent the numbers 1,2,x,y,z,w, by points on the scale.

5. Do the integers form a dense set? Why or why not?

12.27 Summary.

In this chapter we have developed the *rational number system.* In order to see why this system is such an important one, let us retrace some of the steps in its development.

In the *whole number system,* there are two binary operations, addition and multiplication. Subtraction and division are *not* operations. Thus, for example, the subtraction $2 - 5$ and the division $2 \div 5$ are not possible in (W, +, ·). We might say that subtraction and division are "deficiencies" of the whole number system. Part of our work this year has been concerned with removing these deficiencies.

We first removed the subtraction deficiency by developing (Z, +, ·), the number system of *integers.* Subtraction is a binary operation in this system; $2 - 5$, for example, is -3. And since (Z,+, ·) contains an isomorphic copy of (W, +, ·), we have in the integers all of the operations and properties of W, together with the new operation of subtraction. Thus, Z is an "extension" of W, a fact suggested by the following diagram:

$$\begin{tikzcd} \boxed{(W,+,\cdot)} \arrow[r, dashed, "\text{extension to}", "\text{make subtraction an operation}"'] & \boxed{(Z,+,\cdot)} \end{tikzcd}$$

However, division is not an operation on Z, and in this chapter we removed this deficiency by developing the system (Q, +, ·) in which division (except by 0) is always possible. for example, the quotient $2 \div 5$ is the rational number we have called $\frac{2}{5}$. And since

(Q, +, ·) contains an isomorphic copy of (Z, +, ·), Q is an extension of Z. Therefore, we can complete the above diagram as follows:

$$\begin{tikzcd}
\boxed{(W, +, \cdot)} \arrow[r, dashed, "\text{extension to}", "\text{make subtraction an operation}"'] & \boxed{(Z, +, \cdot)} \arrow[d, dashed, "\text{extension division}"', "\text{to make an operation}"] \\
& \boxed{(Q, +, \cdot)}
\end{tikzcd}$$

Is this the only path to follow in removing the subtraction and division deficiencies? The answer is "no," for we might have removed the division deficiency *first.* Thus we could have extended W so that a division such as 2 ÷ 5 became possible. To do this, we could have worked with numbers arising from "positive" fractions, such as those you worked with in elementary school. In this way, we could have obtained a number system in which addition, multiplication, and division (except by 0) were always possible, but not subtraction. If we use (F, +, ·) to denote such a system, the extension can be shown as below:

$$\begin{tikzcd}
\boxed{(W, +, \cdot)} \arrow[d, dashed, "\text{extension division}"', "\text{to make an operation}"] \\
\boxed{(F, +, \cdot)}
\end{tikzcd}$$

Next, we could remove the subtraction deficiency by introducing negatives much as we did in developing the integers in Chapter 4. Then once again we would have arrived at the system (Q, +, ·), as the completed diagram shows:

$$\begin{tikzcd}
\boxed{(W, +, \cdot)} \arrow[d, dashed, "\text{extension division}"', "\text{to make an operation}"] & \\
\boxed{(F, +, \cdot)} \arrow[r, dashed, "\text{extension to}", "\text{make subtraction an operation}"'] & \boxed{(Q, +, \cdot)}
\end{tikzcd}$$

No matter which of the two "paths" is followed, the result is the rational number system (Q, +, ·) in which there are four binary operations – addition, subtraction, multiplication, and division.

In (Q, + , ·). the four operations are defined as follows:

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd} \qquad \frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$$

$$\frac{a}{b} - \frac{c}{d} = \frac{a}{b} + \left(-\frac{c}{d}\right) \qquad \frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} \ (c \neq 0)$$

(Q, +, ·) has the following important properties.

If x, y, and z are rational numbers, then

$$\begin{array}{ll}
(x + y) + z = x + (y + z) & (x \cdot y) \cdot z = x \cdot (y \cdot z) \\
x + 0 = x & x \cdot 1 = x \\
x + (-x) = 0 & x \cdot \frac{1}{x} = 1 \quad (x \neq 0) \\
x + y = y + x & x \cdot y = y \cdot x
\end{array}$$

$$x \cdot (y + z) = (x \cdot y) + (x \cdot z)$$

Any system with two operations which possesses these properties is called a *field.* Therefore, we may speak of the *rational number field,* or the *field of rational numbers.*

The rational number field is ordered. If $\frac{a}{b}$ and $\frac{c}{d}$ are rational numbers, with *b* and *d* both positive, then

$$\frac{a}{b} < \frac{c}{d} \iff ad < bc.$$

The rational number field is *dense.* Between any two different rational numbers, there is another rational number.

12.28 Review Exercises.

1. Solve the following equations.

(a) $4 \cdot x = 3$
(b) $3 \cdot x = 4$
(c) $-4 \cdot x = 3$
(d) $4 \cdot x = -3$
(e) $-4 \cdot x = -3$
(f) $12 \cdot x = 5$
(g) $3 \cdot x = 20$
(h) $3 \cdot x = 21$
(i) $7 \cdot x = 5$
(j) $-3 \cdot x = 8$
(k) $102 \cdot x = 511$
(l) $-55 \cdot x = 30$
(m) $87 \cdot x = 87$
(n) $87 \cdot x = 0$
(o) $4 \cdot x = a$

2. Compute the following.

(a) $\frac{2}{3} + \frac{3}{4}$
(b) $\frac{2}{3} \div \frac{3}{4}$
(c) $\frac{5}{2} - \frac{4}{7}$
(d) $\frac{4}{7} - \frac{5}{2}$
(e) $\frac{8}{5} \cdot \frac{8}{5}$
(f) $\frac{8}{5} \cdot \frac{5}{8}$
(g) $\frac{1}{2} \div \frac{3}{8}$
(h) $\frac{3}{8} \div \frac{1}{2}$
(i) $\frac{9}{4} + \frac{5}{6}$
(j) $\frac{2}{3} \cdot \frac{5}{5}$
(k) $8 - \frac{5}{4}$
(l) $8 \div \frac{5}{4}$
(m) $\frac{5}{4} - 8$
(n) $\frac{5}{4} \div 8$
(o) $3 \div 7$
(p) $7 \div 3$

3. Compute the following.

(a) $(\frac{1}{2} + \frac{3}{4}) + \frac{7}{8}$ (f) $(\frac{3}{8} + \frac{5}{6}) \div \frac{2}{5}$

(b) $\frac{2}{3}(\frac{1}{2} + \frac{3}{5})$ (g) $\frac{3}{8} + (\frac{5}{6} \div \frac{2}{5})$

(c) $\frac{0}{4} + (\frac{9}{5} + \frac{3}{10})$ (h) $(8 \div \frac{1}{3}) - \frac{1}{10}$

(d) $\frac{10}{3} \div (\frac{3}{4} \div \frac{1}{2})$ (i) $\frac{4}{3} \cdot \frac{3}{16} \cdot \frac{3}{4} \cdot \frac{5}{9} \cdot \frac{9}{5} \cdot \frac{16}{3}$

(e) $(\frac{10}{3} \div \frac{3}{4}) \div \frac{1}{2}$ (j) $\frac{-2}{3} + \frac{7}{5} + \frac{-7}{5} + \frac{0}{1} + \frac{2}{3} + \frac{1}{2}$

4. Compute the following:

(a) $\dfrac{\frac{3}{4}}{\frac{7}{8}}$ (c) $\dfrac{\frac{14}{3}}{\frac{7}{5}}$ (e) $\dfrac{\frac{a}{b}}{\frac{c}{d}}$

(b) $\dfrac{\frac{9}{2}}{\frac{2}{9}}$ (d) $\dfrac{\frac{12}{5}}{\frac{3}{8}}$

5. Write each of the following in "expanded notation."

Example: $.23 = (2 \cdot \frac{1}{10}) + (3 \cdot \frac{1}{100})$

(a) .6 (e) 25.08

(b) .63 (f) 3.175

(c) .063 (g) 2.000005

(d) .00603 (h) .3333

6. Write a "decimal fraction" representation of each of the following. If the decimal does not terminate, give an approximation to four decimal places (i.e., correct to ten thousandths).

(a) $\frac{1}{2}$ (f) $\frac{1}{3}$

(b) $\frac{13}{26}$ (g) $\frac{7}{10}$

(c) $\frac{3}{4}$ (h) $\frac{70}{100}$

(d) $\frac{2}{5}$ (i) $\frac{5}{8}$

(e) $3\frac{2}{5}$ (j) $\frac{1}{7}$

7. In each of the following, place one of the three symbols, "<," ">" or "=," so that a true statement results.

(a) $\frac{1}{2}\ \frac{2}{3}$ (d) .3475 .3429 (g) .00001 .000009

(b) $\frac{4}{7}\ \frac{5}{9}$ (e) $\frac{1}{3}$ .333333 (h) $\frac{20}{7}\ \frac{25}{12}$

(c) $\frac{23}{5}\ \frac{25}{7}$ (f) .375 $\frac{3}{8}$ (i) $\frac{-3}{5}\ \frac{-2}{3}$

8. For each pair of rational numbers below, write the name of a rational number that is between them.

(a) $\frac{1}{2}$, 1 (e) $\frac{1}{3}$, $\frac{4}{9}$

(b) $\frac{1}{2}$, $\frac{3}{4}$ (f) .345, .346

(c) $\frac{1}{2}$, $\frac{5}{8}$ (g) $\frac{7}{3}$, $\frac{13}{5}$

(d) $\frac{1}{2}$, $\frac{17}{32}$ (h) 0, $\frac{1}{100}$

(i) 0, .000001

9. Solve the following equations.

(a) $\frac{2}{3} \cdot x = \frac{3}{5}$

(b) $\frac{2}{3} + x = \frac{3}{5}$

(c) $x \cdot \frac{4}{3} = \frac{1}{2}$

(d) $\frac{7}{2} + x = \frac{-4}{5}$

CHAPTER 13: MASS POINTS

13.1 Deductions and Experiments

You have probably noticed that in coming to conclusions we have used two distinct methods. For instance, to convince ourselves that the sum of the measures of the angles of a triangle is 180 (or approximately 180) we can proceed in either of two ways.

(a) We can measure each angle with a protractor and add the measures.

(b) We can show that the statement follows logically from properties of isometries and the parallel property.

The first is an example of the method of *reasoning by induction* in science. It is also used by mathematicians to suggest relations. The second is an example of *reasoning by deduction* and is called *deductive proof* or *mathematical proof.* It shows how one statement follows from others by logical *deductions.*

Many people who are not mathematicians frequently rely on deductions. For instance, a doctor deduces the nature of an illness from symptoms; a surveyor deduces a distance to an inaccessible point from known measurements and mathematical principles; an astronomer deduces the nature of matter in a distant sun from an analysis of the light coming from that sun.

You yourself have surely made deductions. All people do. For instance, when a doorbell is unanswered it is natural to deduce that it is likely that nobody is home.

This chapter differs from other chapters in the sense that in it we allow ourselves proofs by *deduction only.* This will be a novel experience for you, the first of many such experiences in your mathematical studies.

There are many possible systems you can study that will help you to learn about deductive reasoning and its usefulness. We have chosen first the study of *mass points* because of its many applications and its close relation to the geometric ideas you have previously studied.

Naturally, your first question is: What is a mass point? This brings to our attention an important aspect of deductive reasoning which you must try to appreciate before going further. Actually, there are many different objects which are specific interpretations of the general notion of mass points? For instance: a child poised at the end of a see-saw; the earth at a particular position in its orbit; a carbon atom at a particular position inside a complicated molecule.

To establish something of the essential nature of each of these interpretations, we note that in each case a number and a position can be associated. For the child it could be her weight and her position on the see-saw. For the earth it could also be its weight and its position in orbit. For the carbon it could be a number, perhaps its electrical charge, and its location. Each of these cases has the property that a *number* and a *point* are associated. This is what we mean by a mass point.

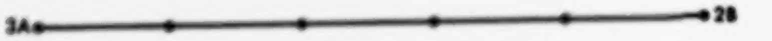
Definition. A mass point is a pair consisting of a positive number and a point.

As you see, different interpretations have some properties in common and some that differ. Faced with such a situation a mathematician lists what he thinks are basic properties common to all and proceeds to make deductions from this list. Since the selected basic properties furnish a beginning in a system they are not deduced. There is nothing in the system from which to deduce them. Such basic property statements are distinguished from those that are deduced. The basic property statements are called *postulates* or *axioms.* Those that are deduced are called theorems.

13.2 Preparing the Way: Notations and Procedures

We need some preparations before stating postulates and deducing theorems. First, it is convenient to have a concise way of referring to a mass point. The mass point with a number 4 at point A will be written "4A". In general the mass point with number *a* at point P will designated "*a*P". If in the course of deduction we conclude that *a*P = *b*Q, this will mean two two things: *a* and *b* name the same number, and P and Q name the same point; that is, *a* = *b* and P = Q. If then A and B name different points then 3A = 2B must necessarily be false; also 4A = 2A must also be false since 4 ≠ 2. We sometimes refer to the number of a mass point at its *weight.*

Second, we illustrate what we mean by *adding* two mass points. It should not be confused with adding two numbers. Suppose 3A and 2B are two mass points, as shown below at points A and B.

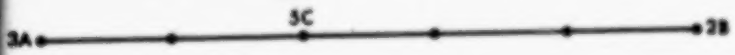


To add them and to represent 3A + 2B as a single point, we must do two things.

(1) Add the weights 3 and 2; 3 + 2 or 5 is the weight of 3A + 2B

(2) Find point C in AB such that AC:CB = 2:3 (Note the *reversal* of 3 and 2 in the ratio 2:3). If on measuring $\overline{AB}$ we find its inch-measure to be 5, AC = $\frac{2}{5} \cdot 5 = 2$ and CB $\frac{3}{5} \cdot 5 = 3$. C is therefore two inches from A and 3 inches from B. C is the *point* in 3A + 2B.

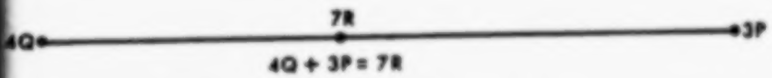
Thus $3A + 2B$ has weight 5 and is at C, or $3A + 2B = 5C$. The sum is represented diagrammatically as follows.



(The equally spaced marks should help you to see that $AC = 2$ and $CB = 3$)

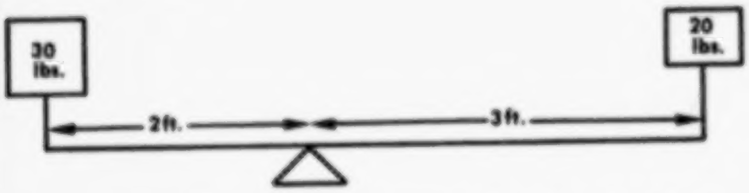
We call C the *center of mass* of the masses at A and B.

Let us consider a second illustration.



Suppose the measure of $\overline{QP}$ in yards is 4. As in the first illustration we find the weight of $4Q + 3P$ to be 7. If R is the center of mass then $QR:RP = 3:4$; that is $QR = \frac{3}{7} \cdot 4$ or $\frac{12}{7}$ and $RP = \frac{4}{7} \cdot 4$ or $\frac{16}{7}$. Thus $QR = 1\frac{5}{7}$ and we can approximate the location of R with a ruler.

The definition for the sum of two mass points is suggested by the see-saw interpretation. Suppose in the diagram below that two weights are placed in the position shown.



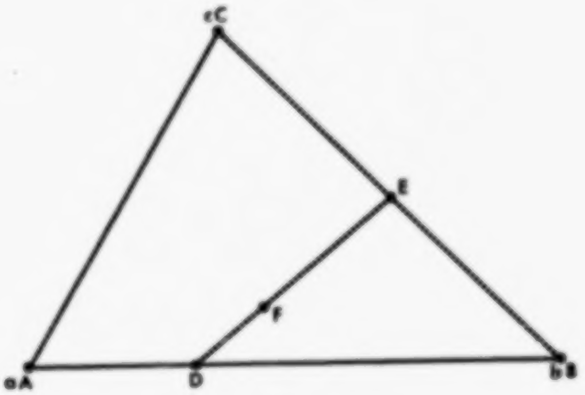
They will be in balance if the weight of each object multiplied by its distance to the balancing point is the same. For data in our diagram the first product is $30 \cdot 2$. The second is $20 \cdot 3$. Are these products the same? If so, the see-saw is in balance.

Compare this situation with the case of the sum of two mass points $30A + 20B$, for which $AB = 5$. The point C, the center of mass, will be $\frac{20}{50} \cdot 5$ feet from A toward B. Is this not the point at which the teeter board balances for the weight 30 and 20 pounds?

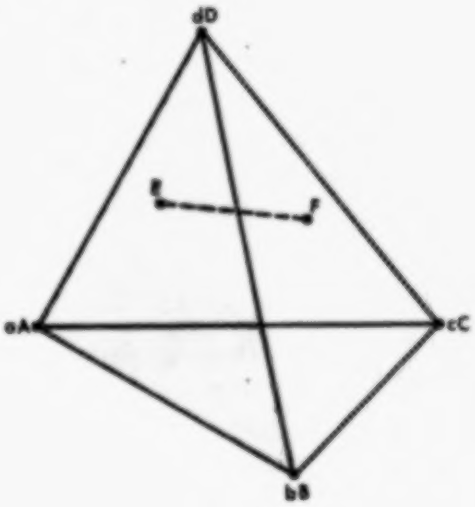
Definition: In general, by $aA + bB$ we shall mean the mass point cC such that $a + b = c$ and C is the point in $\overline{AB}$ such that $AC : CB = b : a$.

In passing we might emphasize that C is *in* $\overline{AB}$. Furthermore, we might guess that each interior point of $\overline{AB}$ can be determined by a correct choice of a and b. Thus, whenever we add *two* mass points, the center of the sum will be found in the segment determined by the mass point addends.

In section 13.4 — 13.8 we will learn to add *three* mass points, not in one line, such as aA, bB, cC shown below. The sum $aA + bB$ must be in $\overline{AB}$, say at D. The sum $bB + cC$ must be in $\overline{BC}$, say at E. Now we have *two* mass points, at D and E and their sum will be in $\overline{DE}$, say at F. F is an interior point of $\triangle ABC$. So the sum of three mass points at non-collinear points determines an interior point of a triangle.



In section 13.12 we add *four* mass points, not in a plane, such as those shown below. Adding three of these determines point in the interior of $\triangle ABC$, $\triangle ABD$, $\triangle BCD$, or $\triangle CAD$. Suppose E is such a point inside $\triangle ABD$, and F is inside $\triangle BCD$. Then, the sum of the mass points at E and F determines a point inside the space figure (a pyramid).



13.3 Exercises

1. In each part below you are given the length of a segment in inches for which you are to draw a diagram. On this diagram represent the sum of the two mass points at a single point.

(a) $AB = 6$, $5A + 1B$
(b) $AB = 6$, $1A + 5B$
(c) $CD = 3$, $2C + D$
(d) $CD = 3$, $1C + 2D$
(e) $EF = 5$, $1E + 1F$
(f) $GH = 3$, $2G + 4H$
(g) $GH = 3$, $3G + 2H$
(h) $KL = 5$, $2K + 4L$
(i) $KL = 5$, $1K + 2L$
(j) $KL = 5$, $1\frac{1}{2}K + 1L$

2. (a) You are given mass points 3A and 4B. Is the

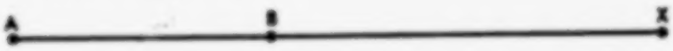
center of mass nearer to A or to B? Try to answer without calculating the position of the center.

(b) Answer the same question for mass points 8A and 5B.

(c) Is the center of masses nearer the point with the greater or lesser weight?

3. For each of the following compute AG:GB.
   - (a) 3A + 2B = 5G
   - (b) 1A + 6B = 7G
   - (c) 2A + 1B = 3G
   - (d) 5A + 5B = 10G

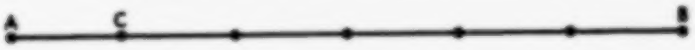
4. In this exercise you are given one of two mass points and the sum. You are to find the other mass point. To illustrate, suppose xX is the missing mass point and 3A + xX = 5B. Thus 3 + x = 5, from which we deduce x = 2. The weight of 3A and xX are 3 and 2. So B is the point in $\overline{AX}$ such that AB:BX = 2:3 and X is in $\overrightarrow{AB}$ with B in between A and X as shown below.



Solve for x and X in each of the following equations.
   - (a) 3A + xX = 4B
   - (b) 4A + xX = 6B
   - (c) xX + 4A = 6B
   - (d) 1A + xX = 3B
   - (e) 2A + xX = 3B
   - (f) xX + 9A = 12B

5. Suppose 12A + bB = cC. What must be true about b and c for each of the following cases?
   - (a) C is the midpoint of $\overline{AB}$.
   - (b) C is the trisection point of $\overline{AB}$ nearer A.
   - (c) C is the trisection point of $\overline{AB}$ nearer B.
   - (d) C is the point of division of $\overline{AB}$ such that AC:BC = 3:4.

6. Draw a line segment $\overline{AB}$ 3 inches long and take C in $\overline{AB}$ such that $\overline{AC}$ is $\frac{1}{2}$ inches long.



   - (a) Represent 1A + 2B at one point. Name it D.
   - (b) Represent 3D + 3C at one point. Name it E.
   - (c) Represent 2B + 3C at one point. Name it F.
   - (d) Represent 1A + 5F at one point. Name it G.
   - (e) Are F and G the same point?
   - (f) If so, how does this exercise show (1A + 2B) + 3C = 1A + (2B + 3C)

7. Let 3 be assigned to A in $\overline{AB}$.
   - (a) If C is the midpoint of $\overline{AB}$, what weight should one assign to B so that C is then the center of mass?
   - (c) If C is the trisection point of $\overline{AB}$ nearer B, what weight should one assign to B so that C is the center of mass?

13.4 Postulates for Mass Points

It is important to know whether addition of mass points is an operation. Otherwise such a sum as 5A + 6B may be assigned more than one mass point and any computation with mass points would become bewilderingly complex. We know that 5A + 6B must have the weight 5 + 6 or 11. But is there exactly one location for the center of mass? It can be proved, with the aid of more mathematics than we have available, that the answer is yes and, moreover, it is between A and B. We shall assume this answer. That is, we accept without proof the statement that there is exactly one mass point for the sum of two mass points. This then becomes our first postulate, the *Closure Postulate*.

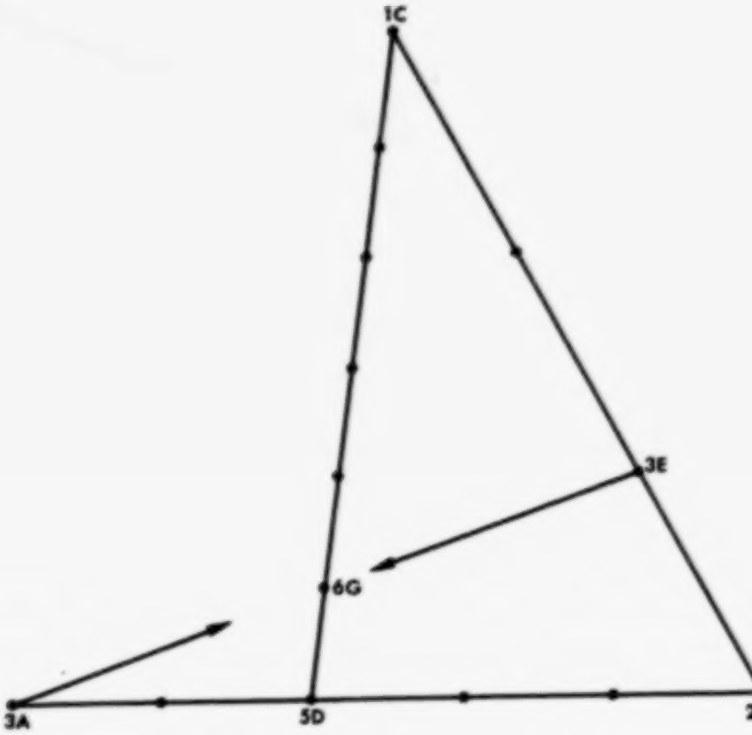
PI. For any two mass points aA and bB there is exactly one mass point cC such that aA + bB = cC.

In effect we are saying that addition of mass points is an operation.

Our construction of aA + bB leads us to accept that aA + bB = bB + aA. We will state this property as a postulate, and call it the *Commutation Postulate* or P2.

P2. For any two mass points aA and bB
aA + bB = bB + aA.

And now we come to a third postulate which we can call the *Association Postulate* or P3. It many not be as obvious as the Closure and Commutation Postulates, and for that reason we shall do an experiment t test its plausibility. We want to see for instance whether (3A + 2B) + 1C = 3A + (2B + 1C), where A, B, C are the points, not necessarily collinear, as shown in this diagram.



To facilitate this experiment we have subdivided $\overline{AB}$ into 5 segments of the same length (3 + 2 = 5) an $\overline{BC}$ into 3 segments of the same length (2 + 1 = 3).

First we find 3A + 2B to be 5D, as shown in the diagram. Then, subdividing $\overline{DC}$ into 6 segments of the same length (5 + 1 = 6) we see (again in the diagram) that 5D + 1C = 6G.

On the other hand we first find 2B + 1C, and find it to be 3E (see the diagram). We have only to test whether 3A + 3E = 6B. To convince ourselves that this is true, or false, we place our ruler on $\overline{AE}$ and see whether G is in $\overline{AE}$ such that AG:GE = 3:3 or 1:1. A test shows it to be true. Try it. We call G the *center of mass* of three masses.

In an exercise you will be asked to further verify by experiment the truth of the Association Postulate.

P3. For all mass points aA, bB, and cC
(aA + bB) + cC = aA + (bB + cC).

This means that aA + bB + cC represents the same mass point no matter how we associate. This mass point has weight a + b + c and its point is the center of mass of the three masses at A, B and C.

We do not claim to have proved the Association property, for we have not *deduced* it. We repeat, the purpose of the experiment is not to prove the property. It is to make it easier to accept as a postulate. (Mathematicians may even accept as postulates statements which cannot be tested as being either true or false.)

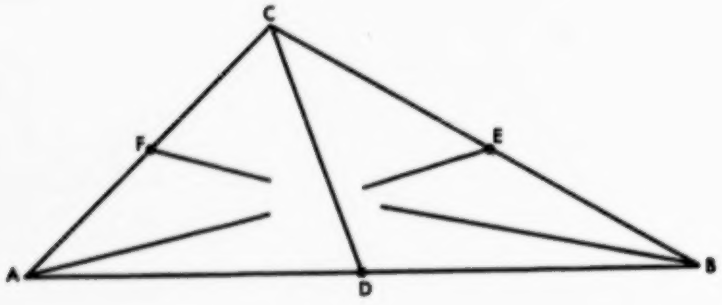
In adding mass points we are also adding positive numbers. It should be understood that we are allowing ourselves to use those properties of (Q, +) which we need. We shall also allow ourselves to use the properties of parallelograms which have appeared earlier in this book.

13.5 Exercises

1. Make an exact copy of the three mass points 3A, 2B and 1C used in the experiment on the preceding page. Show, by an experiment that 3A + 2B + 1C can also be found by any of the following procedures.
   (a) Find 2B + 1C first, then (2B + 1C) + 3A.
   (b) Find 3A + 1C first, then (3A + 1C) + 2B.
2. Justify each of the following statements by citing the appropriate postulate or postulates.
   a. (2B + 1C) + 3A = (1C + 2B) + 3A
   b. (2B + 1C) + 3A = 1C + (2B + 3A)
   c. 2B + 3A + 1C = 3A + 2B + 1C
3. Represent aA + bB + cC in 6 different ways.
4. Make a diagram which shows 2A + 1B + 2C at a single point.
   Take A, B, C as any three noncollinear points.

13.6 A Theorem and a Deduction Exercise

As you recall, we called a statement that is deduced (or is deducible) from other statements a theorem. This, our first theorem for mass points, is about any triangle and may come to you as a surprise. Suppose the triangle is ABC. Let D be the midpoint of $\overline{AB}$, E the midpoint of $\overline{BC}$ and F the midpoint of $\overline{CA}$. Make such a diagram and draw $\overline{CD}$, $\overline{BF}$ and $\overline{AE}$. Do they meet in one point? We shall prove that they do; that is, we shall deduce this from our postulates. To make it easier to talk about the segments $\overline{CD}$, $\overline{BF}$, and $\overline{AE}$, we shall call them medians.



Definition: A segment is a *median* of a triangle if it connects one of its vertices to the midpoint of the side opposite the vertex.

Theorem 1. The three medians of a triangle meet in one point.

To prove this theorem let us start by assigning weights to vertices, thus converting them tomass points. Let us assign 1 to A, 1 to B and also 1 toC. (You will see why we choose 1 as the weight of each point as the proof develops.) We remind you that D is the midpoint of $\overline{AB}$; E is the midpoint of $\overline{BC}$ and F is the midpoint of $\overline{CA}$.

By the Association Postulate (1A + 1B) + 1C = 1A + (1B + 1C). Let us first calculate (1A + 1B) + 1C. First, 1A + 1B = 2D. Then (1A + 1B) + 1C + 2D + 1C. There is a point in $\overline{DC}$, call it G, such that DG : GC = 1:2. Thus 2D + 1C = 3G.

Now we calculate 1A + (1B + 1C). First 1B + 1C = 2E. Then 1A + (1B + 1C) = 1A + 2E. There is a point in AE, call it H, such that AH : HC = 2:1. Then 1A + 2E = 3H. But by the Association Principle 3G = 3H. Therefore G = H, that is, G is the point which divides $\overline{CD}$ in the ratio 2:1 and also the point that divides $\overline{AE}$ in the ratio 2:1.

Now we calculate (1A + 1C) + 1B.

$$\begin{aligned}(1A + 1C) + 1B &= 1A + (1C + 1B) && \text{P3}\\ &= 1A + (1B + 1C) && \text{P2}\\ &= (1A + 1B) + 1C && \text{P3}\\ &= 3G && \text{P1}\end{aligned}$$

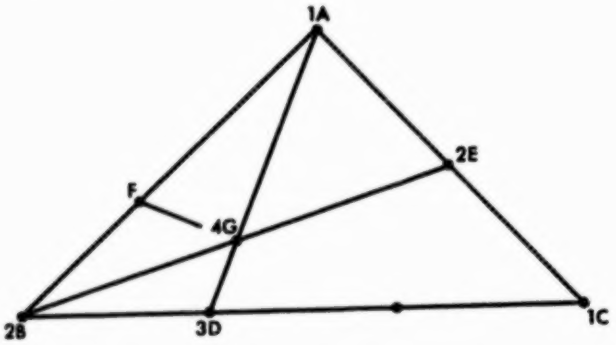
This means that G is also in $\overline{AE}$ and divides it in the ration 2:1.

We have not only proved that the three medians meet in a point (the point G) ,but that this point divides each median in the ratio 2:1 from vertex to midpoint of opposite side.

We can also use postulates to solve problems. This means we will discover theorems. But we won't find it necessary to use these theorems in proving others. Therefore we will not list them formally as theorems. We consider them deduction exercises

Suppose in △ABC, D divides $\overline{BC}$ in the ratio 1:2

from B to C, and E divides $\overline{AC}$ in the ratio 1:1. Let $\overline{AD}$ intersect $\overline{BE}$ in G. What are the numerical values of DG:GA and BG:GE? We can solve this problem as follows. In order that D may be the trisection point of $\overline{BC}$ nearer B, we assign the weights 2 to B and 1 to C. Then 2B + 1C = 3D. In order that E be the midpoint of CA we assign the same weight to A as to C. Having assigned 1 to C we assign 1 to A also. Then 1C + 1A = 2E. The point of (2B + 1C) + 1A is the same as the point of 2B + (1C + 1A). This point is on $\overline{AD}$ and $\overline{BE}$; that is, this point is the intersection of $\overline{AD}$ and $\overline{BE}$, and it is named G. Therefore (2B + 1C) + 1A = 3D + 1A = 4G, and thus DG:GA = 1:3. Also 2B + (1C + 1A) = 2B + 2E = 4G, and thus BG:GE = 1:1.



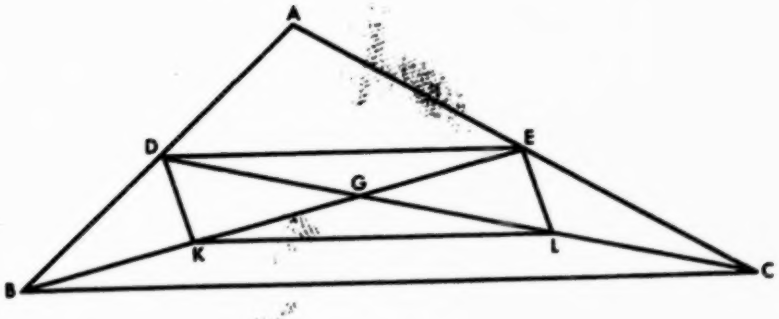
We can extend our discoveries in this problem. Let $\overrightarrow{CG} \cap \overline{AB}$ = F. By P2 and P3, (2B + 1A) + 1C = 4G. Therefore 2B + 1A is a mass point whose center is in $\overline{BA}$ and also on $\overrightarrow{CG}$. It can be only F. Thus 2B + 1A = 3F and BF:FA = 1:2. From 3F + 1C = 4G, it follows that FG:GC = 1:3.

If we omit explanations, the solution of the above problem can be written briefly as follows

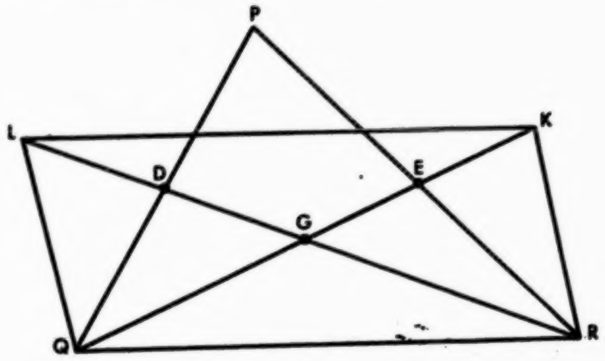
1. 2B + 1C = 3D and 3D + 1A = 4G. Therefore DG:GA = 1:3.
2. 1C + 1A = 2E and 2B + 2E = 4G. Therefore BG:GE = 1:1.
3. 2B + 1A = 3F Therefore BF:FA = 1:2.
4. 3F + 1C = 4G Therefore FG:GC = 1:3.

13.7 Exercises

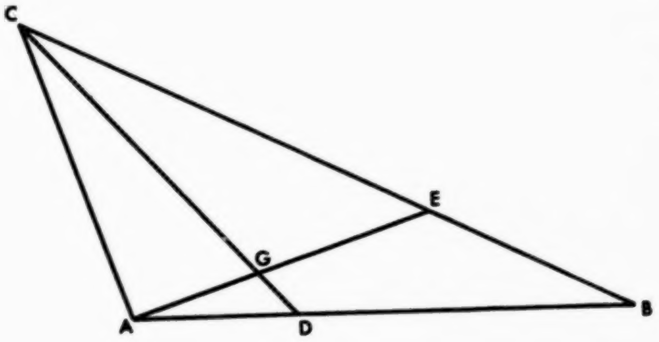
1. Review the proof of the theorem about the median of a triangle, then tell whether you think the proof applies only to the triangle represented in the diagram or to all triangles.
2. This is an experiment exercise. Draw any triangle, locate the midpoint of each side and draw the medians. In your diagram, do the medians meet at one point? Suppose they did not, or they did not, in a drawing made by a classmate. Try to find why the drawing does not agree with the theorem.
3. The lengths of the medians of a triangle are 15, 12 and 18 inches long. How long are the segments into which each median is divided by the point in which they meet?
4. Answer the question in Exercise 3 if the medians are 12, 13, 14 inches long.
5. In ΔABC, $\overline{CD}$ and $\overline{EF}$ are medians, meeting at G. K is the midpoint of $\overline{BG}$ and L is the midpoint of $\overline{CG}$. Prove (by deduction, of course) that DELK is a parallelogram.



6. In ΔPQR, $\overline{QE}$ and $\overline{RD}$ are medians, meeting at G. D is the midpoint of $\overline{LG}$ and E is the midpoint of $\overline{KG}$. Prove: LKRQ is a parallelogram.



7. For the data in Exercise 6, prove: LK = QR and LQ = KR.
8. In ΔABC, D is in $\overline{AB}$ and AD:DB = 1:2. E is in $\overline{BC}$ and BE:EC = 1:2. Let $\overline{AE} \cap \overline{CD}$ = G.



Prove : AG:GE = 3:4
CG:GD = 6:1

(Hint: Assign weight 4 to A, 2 to B and 1 to C.)

9. Using the data in Exercise 8 let $\overline{BG} \cap \overline{CA}$ = F and find the numerical value of BG:GF and AF:FC.
10. Add to the data in Exercise 8 that K is in $\overline{CA}$ and CK:KA = 1:2. Let $\overline{BK} \cap \overline{AE}$ = L and $\overline{BK} \cap \overline{CD}$ = M. Prove: BL = LM = 3MK (This is a difficult exercise).

13.8 Another Theorem

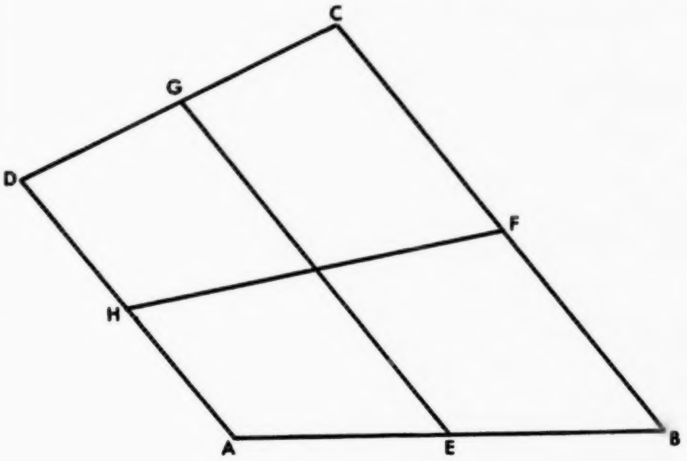
Our definition for addition over mass points applies to two mass points. In other words, addition is a binary operation. To make it possible to add three mass points we introduced the Association Postulate, which says that aA + bB + cC can be found by either finding (aA + bB) first or (bB + cC) first.

Either of these sums can be found and then a second addition completes the calculation by which $aA + bB + cC$ is expressed as a mass point with one weight and one point. For our next theorem we need to know how to add four mass points. This can be done by a repeated application of the Association Postulate, as follows:

$aA + bB + cC + dD = (aA + bB) + (cC + dD)$. There are also other ways to associate. For instance, $aA + (bB + cC) + dD$. This reduces the addition from four to three mass points. And now a second theorem.

Theorem 2. The segments joining the midpoints of opposite sides of a quadrilateral bisect each other.

Proof: Let ABCD be the quadrilateral and let E be the midpoint of $\overline{AB}$, F the midpoint of $\overline{CD}$ and H the midpoint of $\overline{DA}$. We have to prove that $\overline{EG}$ bisects $\overline{HF}$.



We assign the wieght 1 to each of A, B, C, D. then we have the following equations:

(1) $1A + 1B = 2E$
(2) $1B + 1C = 2F$
(3) $1C + 1D = 2G$
(4) $1D + 1A = 2H$.

By P3 and P2 we can show that

$$(1A + 1B) + (1C + 1D) = (1D + 1A) + (1B + 1C).$$

Thus $2E + 2G = 2H + 2F$.

If K is the midpoint of $\overline{EG}$ then $2E + 2G = 4K$.

If L is the midpoint of $\overline{HF}$ then $2H + 2F = 4L$.

Thus $4K = 4L$.

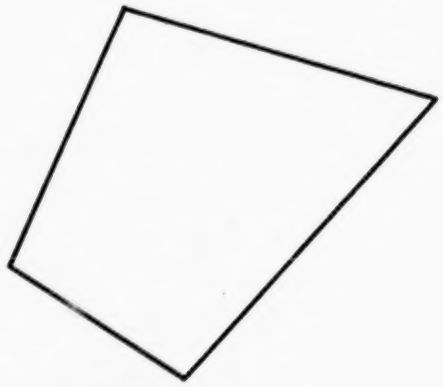
or $K = L$

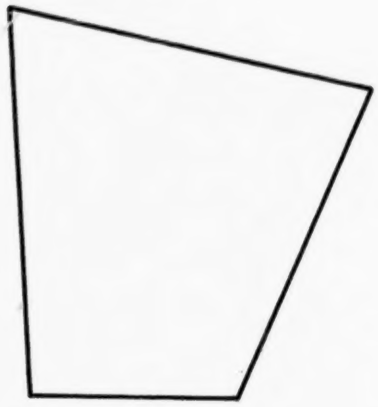
Do you see that this completes the proof?

Incidentally, what kind of figure is EFGH? State another theorem that follows immediately from the one we just proved.

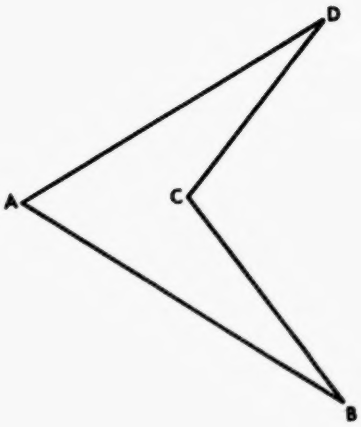
13.9 Exercises

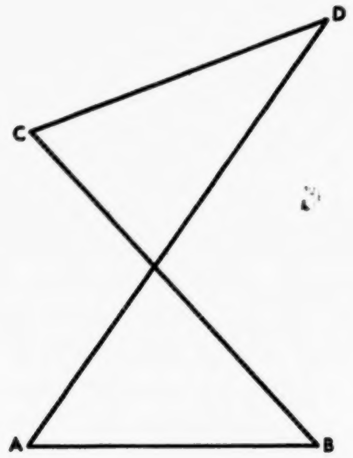
1. The purpose of this exercise is to see if an experiment agrees with Theorem2. In performing the experiment you should be careful to draw straight lines and to locate midpoints accurately. Perform the experiment on two different quadrilateral figures having shapes such as the ones suggested by the following diagrams.



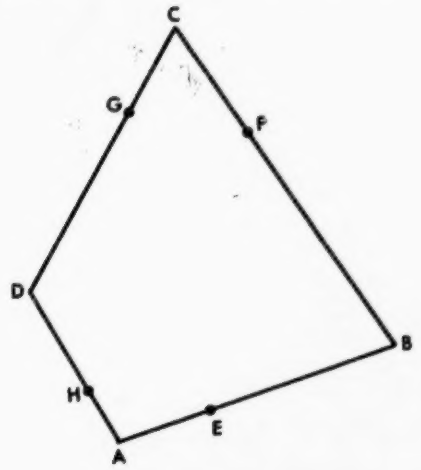


2. Verify whether the theorem is true for such figures as those below. They are named ABCD to tell you that the sides $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DA}$, in that order. This means that $\overline{AB}$ and $\overline{CD}$ are a pair of opposite sides and $\overline{BC}$ and $\overline{DA}$ are another pair of opposite sides.

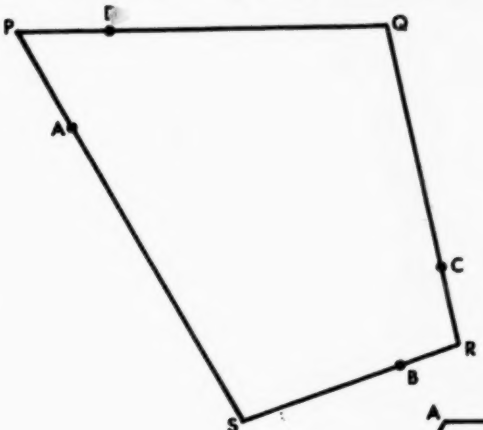


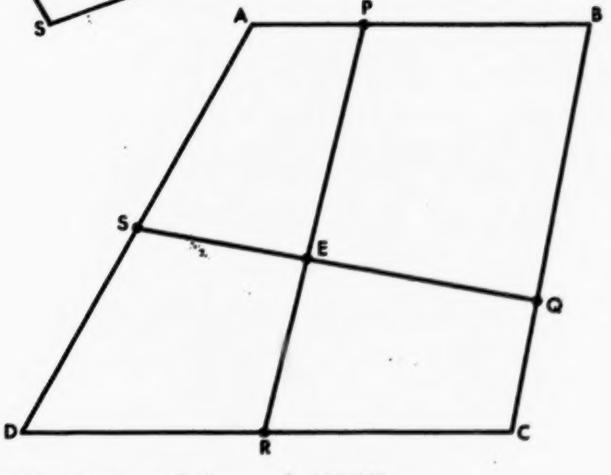


3. In the quadrilateral ABCD shown
   AR:EB = 1:2, BF:FC = 2:1
   CG:GD = 1:2 and DH:HA = 2:1.
   Prove: EG and FH bisect each other. (Hint: Assign weights 2 to A, 1 to B, 2 to C and 1 to D.)



4. In the quadrilateral PQRS shown
   PA:AS = 1:3, SB:BR = 3:1,
   RC:CQ = 1:3, QD:DP = 3:1 as shown.
   Prove: $\overline{AC}$ and $\overline{BD}$ bisect each other.



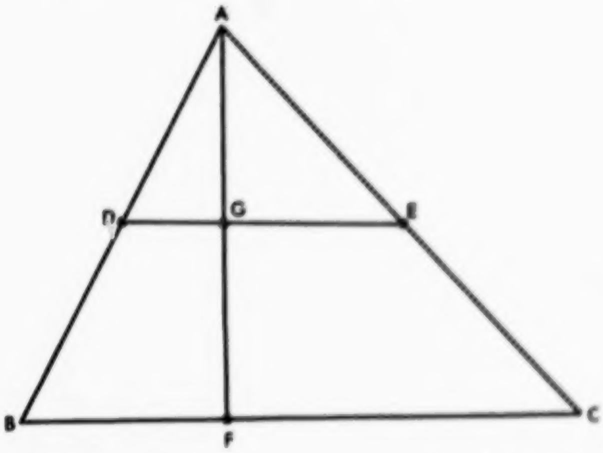


5. As shown for the quadrilateral ABCD,
   AP:PB = 1:2, BQ:QC = 2:1, CR:RD = 1:1,
   DS:SA = 1:1. Let $\overline{SQ} \cap \overline{PR}$ = E. Find the numerical values of RE:EP and SE:EQ.

13.10 A Fourth Postulate

Before introducing the fourth postulate let us examine a problem which requires this postulate.

In △ABC, D is the midpoint of $\overline{AB}$, E is the midpoint of $\overline{AC}$, and F is the trisection point of $\overline{BC}$ nearer B. Let $\overline{DE} \cap \overline{AF} = G$. We are required to show that G is the midpoint of $\overline{AF}$ and also the trisection point of $\overline{DE}$ nearer D.



We begin by assigning a weight of 1 to C. In order that F be the trisection point of $\overline{BC}$ nearer B we assign 2 to B. Thus 2B + 1C = 3F.

Let us now consider what weight to assign to A. First, in order that D be the midpoint of $\overline{AB}$ we should assign to A the same weight that we assigned to B, that is, 2. In order that E be the midpoint of $\overline{AC}$ we should assign to A the same weight that we assigned to C, that is 1. Thus we find ourselves assigning two weights to A, or to put it another way, at A we are to have two mass points at one point; one is 2A, the other is 1A. If we could add these two mass points we could then complete the solution. But our definition for addition of two mass points applies to two mass points at different locations. So we must agree on how to add 2A and 1A. Before we make a formal statement on how to add them, you might wish to suggest a method. But whatever the method, it will be a postulate, and we call it P4.

P4. For all positive numbers a and b and all points P

$$aP + bP = (a + b)P.$$

By this postulate 2A + 1A = 3A.

To continue with our solution, we note that 2B + 1C + 3A can be calculated either as (1) (2B + 1C) + 3A, or as (2) (2A + 2B) + (1A + 1C). Since 2B + 1C = 3F, (1) becomes 3F + 3A which is equal to 6 H where H is in FA such that FH:HA = 1:1.

Since 2A + 2B = 4D and (1A + 1C) = 2E, (2) becomes 4D + 2E which is equal to 6K, where K is in $\overline{DE}$ such that DK:KE = 1:2. But whether we calculate 2B + 1C + 3A either way we get the same result. Thus 6H = 6K or H = K. Since H is on both $\overline{FA}$ and $\overline{DE}$, H = $\overline{FA} \cap \overline{DE}$ = G.

The actual calculations are few and can be written briefly as follows.

$2B + 1C + 3A$ is equal to

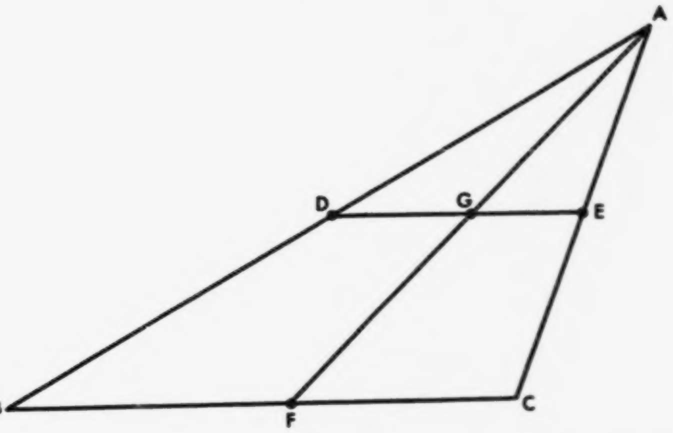
$(2B + 1C) + 3A$ or $(2A + 2B) + (1A + 1C)$
$3F + 3A$ | $= 4D + 2E$
$6H$ | $= 6K$

Therefore $H = K = G$.

us $FG:GA = 1:1$ $DG:GE = 1:2$

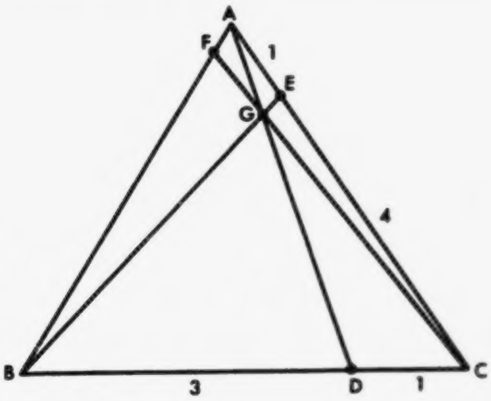
1 Exercises

uppose in $\triangle ABC$, D is the midpoint of $\overline{AB}$ and E
s the midpoint of $\overline{AC}$, and
is in $\overline{BC}$ such that $BF:FC = 5:4$
nd $\overline{DE} \cap \overline{AF} = G$.
rove: G is the midpoint of $\overline{AF}$
$DG:GE = 5:4$
Hint: Assign 4 to B and 5 to C)



State a theorem which seems to be suggested by Exercise 1 and the problem of section 13.10. Investigate the case in which we take trisection points of $\overline{AB}$ and $\overline{AC}$, both nearer A, instead of the midpoints.

In $\triangle ABC$, D is in $\overline{BC}$ and $\frac{BD}{DC} = \frac{3}{1}$, E is in CA and $\frac{CE}{EA} = \frac{4}{1}$, and F is in $\overline{AB}$. $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$ meet at point G.



(a) Find $\frac{AF}{FB}$

(b) Prove: $\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = 1$.

(Hint: Assign 1 to B. What should you assign then to C? Then to A?)

5. Suppose in Exercise 4 $\frac{BD}{DC} = \frac{3}{2}$ and $\frac{CE}{EA} = \frac{5}{3}$.

*6. Exercises 4 and 5 are special cases of a theorem called Ceva's theorem, named after an Italian who is said to have discovered it. Ceva's theorem says: In $\triangle ABC$, if D, E, F are interior points of AB, BC and CA respectively and AD, BE, and CF meet in one point then

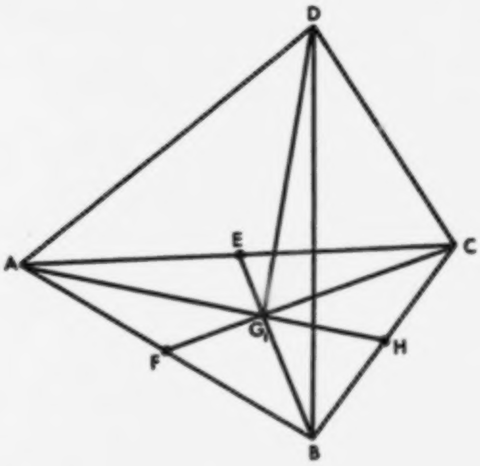
$$\frac{AF}{FB} \bullet \frac{BD}{DC} \bullet \frac{CE}{EA} = 1$$

Try to prove it. (Hint: Let $BD = a$, $DC = b$, $CE = c$, $EA = d$) (Difficult).

*7. For the data in Ceva's Theorem prove $\frac{GD}{AD} + \frac{GE}{BE} + \frac{GF}{CF} = 1$, where G is the point in which $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$ meet (Difficult).

13.12 A Theorem in Space

At the beginning of this chapter we worked with mass points at points on a line. Then we went on to work with mass points in a plane. We end this chapter with a theorem about points in space.



We begin with four points, A, B, C, and D not in a plane (see the figure). Let us look at $\triangle ABC$ and its medians $\overline{AH}$, $\overline{BE}$, and $\overline{CF}$. We know from Theorem 1 that these medians meet in a point, name it G. The point in which the medians of a triangle meet is called the *centroid* of the triangle. In what ratio does the centroid G divide $\overline{AH}$, from A to H? Now, $\triangle BCD$, $\triangle ABD$, and $\triangle ADC$ also have centroids. Consider the segments joining the centroid of one of these triangles to the fourth point. One such segment is $\overline{GD}$ since it joins the centroid of $\triangle ABC$ to D. How many such segments are there? Do you think that these four segments meet at a point? Indeed they do and that is what our space theorem says.

Theorem 3. If A, B, C, D are points in space, not in a plane, and G_1 is the centroid of $\triangle ABC$, G_2 is the centroid of $\triangle DAB$, G_3 is the centroid of $\triangle DBC$ and G_4 is the centroid of $\triangle DCA$, then $\overline{DG_1}$, $\overline{CG_2}$, $\overline{AG_3}$, and $\overline{BG_4}$ meet in a point which divides each of these segments in the ratio 1:3 from centroid to the point.

To prove this theorem we assign weight 1 to each of A, B, C, D. Then we consider $1A + 1B + 1C + 1D$.

One way to calculate this is to associate $(1A + 1B + 1C)$ which is $3G_1$. Then $3G_1 + D = 4H$, where H is a point in $\overline{G_1D}$ such that $G_1H : HD = 1:3$. Thus $1A + 1B + 1C + 1D = 4H$, and whether we calculate it as $(1A + 1B + 1D) + 1C$, or $(1B + 1C + 1D) + 1A$, or $(1A + 1C + 1C + 1D) + 1B$, we continue to get 4H. Do you see that this completes the proof?

13.13 Chapter Summary

In this chapter we studied some properties of mass points deductively. We started by defining mass points and addition of mass points. The first postulate (closure) assured us that this addition is an operation. The second and third provide the properties of commutation and association. Later we added a fourth postulate that enables us to add two weights when they are assigned to the same point. We deduced three statements which you may find useful to remember. We labeled them theorems. One claims that the medians of a triangle meet in a point. Another claims that the segments joining midpoints of opposite sides of a quadrilateral bisect each other. The third is about four points in space, not in a plane, and the centroids of the four triangles determined by each triple of the four points. It claims that the segments joining the centroid of each triangle to the fourth point meet in a point that divides each segment in the ratio 1:3 from the centroid to the point.

But we also solved many exercises by deductions and thus proved many statements which we did not dignify by calling them theorems, even though they are theorems, because we probably won't find them useful in proving other theorems.

The most important aspect of this chapter is the prodcure of deducing theorems from postulates.

13.14 Review Exercises

1. Draw $\overline{AB}$ making it 3 inches long. Let C be its midpoint. Locate the center of masses for the following mass points.

   (a) $2A + 1B$
   (b) $1A + 2B$
   (c) $2A + 1C$
   (d) $1A + 1B + 1C$
   (e) $A + 2C + 3B$
   (f) $2A + 4B + 3C$.

2. Solve for x and locate x in a drawing of $\overleftrightarrow{AB}$ where $\overline{AB}$ is a one inch segment.

   (a) $3A + xX = 4B$
   (b) $2A + xX = 3B$
   (c) $xX + 2A = 4B$
   (d) $xX + 3A = 5B$.

3. Let A have weight 8 and let $\overline{AB}$ be a given segment. Let C be the center of mass for masses at A and B. What weight should you assign B for each of the following descriptions of C.

   (a) C is the midpoint of $\overline{AB}$.

   (b) C is the bisection point of $\overline{AB}$ nearer A.

   (c) C is the trisection point of $\overline{AB}$ nearer B.

   (d) C is the point of $\overline{AB}$ such that $AC:CB = 2:3$

4. In $\triangle ABC$, D is the midpoint of $\overline{BC}$ and E is the point in $\overline{CA}$ such that $CE:EA = 4:1$.

   (a) If 1 is assigned to B, what should you assign to C and A so that D is the center of masses at B and C, and E is the center of masses at C and A?

   (b) If $\overline{AD} \cap \overline{BE} = G$, compute the values of $AG:GD$ and $BG:GE$.

   (c) If $\overleftrightarrow{CG} \cap \overline{AB} = F$, compute $AF:FB$.

5. In $\triangle ABC$, D is in $\overline{AB}$ and $AD:DB = 1:2$: E is $\overline{BC}$ and $BE:EC = 2:1$. F is in $\overline{CA}$ and $CF:FA = 1:2$. Prove that $\overline{DF}$ and $\overline{AE}$ bisect each other.

6. In quadrilateral ABCD, E, F, G, H, are respectively in $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DA}$. Each of $AE:EB$, $BF:FC$, and $CG:GD$ is equal to 2:1, $DH:HA = 1:8$, and $\overline{EG} \cap \overline{FH} = K$. Prove $EK:KG = 4:1$ and $FK:KH = 3:2$.

CHAPTER 14

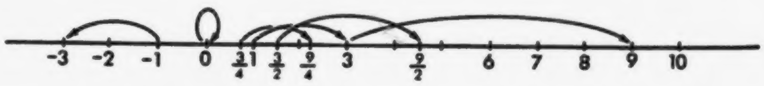
SOME APPLICATIONS OF THE RATIONAL NUMBERS

14.1 Rational Numbers and Dilations.

In Chapter 7, you learned that "D_{ab}" means "$D_b \circ D_a$," the dilation D_a followed by the dilation D_b, at that time, it was required that *a* and *b* be integers. Let us now consider the composition $D_b \circ D_a$, where *a* and *b* are rational numbers. We shall restrict the discussion to dilations on a line. In the exercises, dilations in the plane will be considered. In particular, let us start with

$$D_{\frac{1}{2}} \circ D_3.$$

Since D_3 acts first, we show below the images of certain points under this dilation.

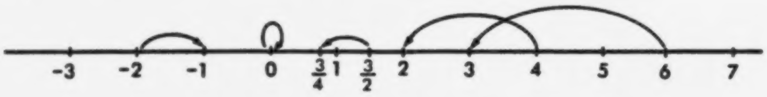


Since we now have the rational numbers, any point with a rational coordinate has an image under this dilation. For instance, the point with coordinate $\frac{3}{4}$ is mapped into the point with coordinate $\frac{9}{4}$, since $3 \cdot \frac{3}{4} = \frac{9}{4}$.

Question: Under the dilation D_3, what are the coordinates of the images of the points having the following coordinates?

$\frac{1}{3}$; 1; $\frac{2}{3}$; 10; 100; -1; $-\frac{1}{3}$.

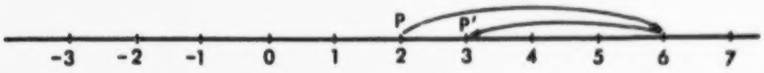
How shall we interpret $D_{\frac{1}{2}}$? In order to be consistent with the way in which we interpreted D_2, where *a* is an integer, we shall say that under $D_{\frac{1}{2}}$ a point P is mapped into a point P' whose distance from the origin is $\frac{1}{2}$ times the distance of P from the origin. The images of certain points under the dilation $D_{\frac{1}{2}}$ are shown below.



Question: Under the dilation $D_{\frac{1}{2}}$, what are the coordinates of the images of the points having the following coordinates

1; 2; $\frac{1}{2}$; $\frac{3}{2}$: 10: 100: -2.

We are now ready to consider the composition $D_{\frac{1}{2}} \circ D_3$. The diagram below shows the image (under this composition) of the point with coordinate 2.



Do you see that under the composition $D_{\frac{1}{2}} \circ D_3$, and point P has an image P' whose distance from the origin is $\frac{3}{2}$ times the distance of the point P from the origin. In other words, we may write:

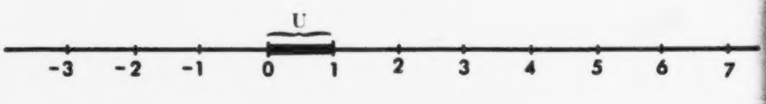
$$D_{\frac{1}{2}} \circ D_3 = D_{\frac{3}{2}}.$$

Thus we see that the dilation $D_{\frac{3}{2}}$ may be considered as the composition of two dilations.

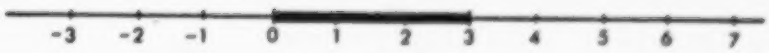
Question: Since under $D_{\frac{3}{2}}$ the image of any point is $\frac{3}{2}$ as far from the origin as the point itself, what do you think the *inverse* of $D_{\frac{3}{2}}$ is?

Question: Can you express $D_{\frac{3}{4}}$ as the composition of two dilations?

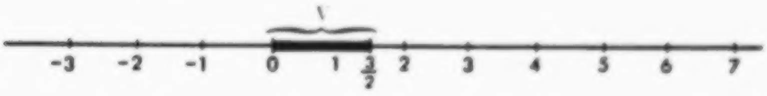
It is also instructive to look at what happens to a *segment* under a dilation such as $D_{\frac{3}{2}}$. In particular, let us look at the segment whose endpoints are those having coordinates 0 and 1; such a segment is often called a *unit segment*, and we shall denote it by "U."



Now since $D_{\frac{3}{2}}$ is the composition $D_{\frac{1}{2}} \circ D_3$, do you see that segment U is first "stretched" to



a segment 3 times as long. Then, that segment is "shrunk" to a segment half as long, as the



agrams show. The final segment, which has been beled V, is then the image of U under the dilation $\frac{3}{2}$. We may simply write which may be red "V is

$$V = \frac{3}{2}U,$$

times U,'' or "V is $\frac{3}{2}$ of U." This means that the

ngth of segment V is $\frac{3}{2}$ times the length of segment

$(\frac{3}{2} \cdot 2 = 3.)$

Example 1. If a segment X has a length of 10 inches, what is the length of $\frac{3}{4}X$?

We could think of this problem in terms of the dilation $D_{\frac{3}{4}}$ on a line.

If the segment X is first "stretched" by 3, the resulting segment has a length of 30 inches. If that segment is then "shrunk" by $\frac{1}{4}$. the length of the resulting segment is $\frac{1}{4} \cdot 30$, or $\frac{30}{4}$ inches.

In practice, of course, it is not necessary to explain the solution in this way. We may simply write

$$\frac{3}{4} \text{ of } 10 = \frac{3}{4} \cdot 10 = \frac{30}{4} \left(\text{or } \frac{15}{2}\right).$$

Example 2. If segment X has length 10 inches, what is the length of $\frac{4}{3}X$?

$$\frac{4}{3} \text{ of } 10 = \frac{4}{3} \cdot 10 = \frac{40}{3}.$$

Hence, the length of $\frac{4}{3}X$ is $\frac{40}{3}$ inches.

Notice that in Example 1 the final segment is shorter an the segment X, while in Example 2 the final segnt is longer than X. Is there any way to predict this forehand from the dilations $D_{\frac{3}{4}}$ and $D_{\frac{4}{3}}$? (Compare the "stretcher" and "shrinker" in each case.)

Question: How must a and b be related so that under the dilation $D_{\frac{a}{b}}$,

1) the image of a segment is longer than the segment itself?

2) the image of a segment is shorter than the segment itself?

3) the image of a segment is the segment itself?

14.2 Exercises.

1. Draw three separate number scales, and on each mark points with the following coordinates:

   $0, 1, 2, \frac{5}{2}, \frac{3}{4}$ and -1.

   (a) On one of the drawings, show the image of each of the points under the dilation D_2.

   (b) On another of the drawings, show the image of each of the *images* from part (a) under the dilation $D_{\frac{1}{3}}$.

   (c) On the third drawing, show the images of each of the original points under the composition $D_{\frac{1}{3}} \circ D_2$.

   (d) Express the composition of dilations in part (c) as a single dilation.

   (e) Express each of the following as single dilations D_x, where x is a rational number:

   $D_{\frac{1}{5}} \circ D_4$: $D_{\frac{1}{3}} \circ D_7$: $D_{\frac{1}{2}} \circ D_{10}$: $D_{10} \circ D_{\frac{1}{2}}$.

2. Draw two number scales, and on each mark points with the following coordinates:

   (b) On another drawing, show the image of each of the original points under the dilation $D_{\frac{2}{4}}$.

   $0, 1, 2, 3, 4, \frac{1}{2}, \frac{8}{3}$ and -2.

   (a) On one drawing, show the image of each of these points under the dilation $D_{\frac{1}{2}}$.

   (c) Is it correct to write: $D_{\frac{1}{2}} = D_{\frac{2}{4}}$?

   (d) When is $D_{\frac{a}{b}} = D_{\frac{c}{d}}$?

3. On a number scale, let P be the point with coordinate 2.

   (a) Let P' be the image of P under $D_{\frac{5}{3}}$. What is the coordinate of P'?

(b) Let P'' be the image of P' under $D_{\frac{2}{3}}$. What is the coordinate of P''?

(c) What is the image of the original point P under the composition $D_{\frac{2}{3}} \circ D_{\frac{5}{3}}$?

(d) Can you write the composition in part (c) as a single dilation?

4. (a) Write a single dilation $D_{\frac{x}{y}}$ for the composition $D_{\frac{7}{3}} \circ D_{\frac{5}{2}}$.

(b) According to the definition we made in Chapter 12, what is the product $\frac{7}{3} \cdot \frac{5}{2}$?

In this section, we have used dilations to give meaning to a statement such as "$\frac{3}{2}$ of X," where X is a segment. And this kind of expression is common in everyday uses of mathematics. For example, if X represents a *class of students*, then "$\frac{2}{3}$ of X" (that is, "$\frac{2}{3}$ of the class") can be interpreted in much the same way as with segments. We really mean $\frac{2}{3}$ times the measure of X. And in this case, the measure is a whole number (size of a set). Thus, if there are 30 people in the class, "$\frac{2}{3}$ of the class" is 20, since $\frac{2}{3} \cdot 30 = 20$. Problems 5 through 12 are of this kind.

5. There are 100 senators in the United States Senate. On a recent vote, $\frac{13}{20}$ of the Senate voted "yes" on a certain bill. How many Senators voted "yes"?

6. A certain state has an area of 70,000 square miles. $\frac{3}{100}$ of the state is irrigated land. How many square miles in the state are irrigated?

7. Jim has $2000 in the bank, and the bank is supposed to pay him $\frac{3}{100}$ of that amount for interest. How much should Jim receive?

8. In 1960, the population of a certain town was 18,000. Today the population is $\frac{5}{3}$ of that number. What is the population today?

9. A family spends $\frac{23}{100}$ of its income on food. If the income for one year is $8500, how much money does this family spend for food in one year?

10. If one pound of ground meat costs $.90 what will be the cost of $2\frac{1}{2}$ pounds?

11. (a) If Jim's height is $\frac{4}{3}$ of Bill's height, who is taller?

(b) If Mary's height is $\frac{3}{4}$ of Sue's height, who is taller?

(c) If Bob's height is $\frac{4}{4}$ of John's height, who is taller?

12. In a certain town, there are 5000 registered voters. And, a recent election, 3500 people voted. What "fraction" of the town's registered voters actually voted? (Express your answer by an irreducible fraction $\frac{a}{b}$. Check your result by showing that $\frac{a}{b}$ of 5000 is 3500.)

13. In this problem we consider dilations D_x, where x is a rational number, in the *plane*. Just as Z × Z is the lattice of all points with coordinates (a,b), where a and b are integers, so Q × Q is the lattice of all points with coordinates (x,y), where x and y are rational numbers.

(a) Draw a pair of axes, and plot all points whose coordinates are (a,b), where a and b are integers between −4 and 4.

(b) Now plot a point with coordinates $(\frac{3}{2}, \frac{7}{4})$. Note that this point does not belong to Z × Z, but it does belong to Q × Q.

(c) Consider the dilation D_2. Under this dilation, the image of $(\frac{3}{2}, \frac{7}{4})$ is defined to be $(2 \cdot \frac{3}{2}, 2 \cdot \frac{7}{4})$, or $(3, \frac{7}{2})$. Plot this image point. (Do you see a segment in the plane that has been "stretched" to twice its original length?)

(d) Under the dilation $D_{\frac{1}{2}}$, the image of $(\frac{3}{2}, \frac{7}{4})$ is $(\frac{1}{2} \cdot \frac{3}{2}, \frac{1}{2} \cdot \frac{7}{4})$. Plot this image point. (Do you see a segment in the plane that has been "shrunk" to $\frac{1}{2}$ of its original length?)

14. From Exercise 13, we make the following definition: If (x,y) is an element of Q × Q, and D_c is a dilation where c is a rational number, then the image of (x,y) under D_c is (cx,cy).

(a) Plot the images of the following points under $D_{\frac{3}{4}}$: (2,8): (4,12): (9, −4): (−8,6): (−2,−12): (0,0): (1,1).

(b) Now for each *image* from part (a), plot the image of that image under $D_{\frac{4}{3}}$.

(c) How are the dilations $D_{\frac{3}{4}}$ and $D_{\frac{4}{3}}$ related?

15. (a) How would you describe the images of the points in $Q \times Q$ under the dilation D_0?

    (b) How would you describe the images of the points in $Q \times Q$ under the dilation D_1?

    (c) How would you describe the images of the points in $Q \times Q$ under the dilation D_{-1}?

14.3 Computations with Decimal Fractions.

In Section 14.1 we dealt with such problems as that of finding "$\frac{3}{4}$ of X." For example if X is a segment having length $2\frac{1}{2}$ inches, then

$$\frac{3}{4} \text{ of } X = \frac{3}{4} \cdot 2\frac{1}{2} = \frac{3}{4} \cdot \frac{5}{2} = 1\frac{7}{8}.$$

At times, problems such as this are expressed in terms of decimal fractions. For instance, we could just as easily speak of finding .75 of a segment X whose length is 2.5 inches. Then we would have to compute

$$.75 \times 2.5.$$

The result should be the same as before, $1\frac{7}{8}$. How is the computation with decimal fractions carried out? Study the computation below.

$$.75 \times 2.5 = \frac{75}{100} \times \frac{25}{10} = \frac{1875}{1000} = 1.875$$

Thus, $.75 \times 2.5 = 1.875$.

This computation could be done as below

$$\begin{array}{r} 2.5 \\ \underline{\times\ .75} \\ 125 \\ \underline{175\ } \\ 1.875 \end{array}$$

There is a relationship between the number of digits to the right of the decimal place in the product 1.875, and the number of digits to the right of the decimal point in the two factors, 2.5 and .75. Do you see what the relationship is? (it is a result of the fact that $100 \times 10 = 1000$.)

Question: To which of the following is the product 1.5×1.5 equal?

.225: 2.25: 22.5: 225.

What is the sum of \$2.45 and \$3.87? The computation is shown below.

$$\begin{array}{r} \$2.45 \\ \underline{+\ \$3.87} \\ \$6.32 \end{array}$$

Notice that we "add tenths to tenths, hundredths to hundredths," etc.

$$2.45 = 2 + \frac{4}{10} + \frac{5}{100}; \quad \text{and}$$

$$3.87 = 3 + \frac{8}{10} + \frac{7}{100}.$$

Then,

$$\begin{aligned} 2.45 + 3.87 &= (2 + \frac{4}{10} + \frac{5}{100}) + (3 + \frac{8}{10} + \frac{7}{100}) \\ &= (2+3) + (\frac{4}{10} + \frac{8}{10}) + (\frac{5}{100} + \frac{7}{100}) \\ &= 5 + \frac{12}{10} + \frac{12}{100} \\ &= 5 + \frac{13}{10} + \frac{2}{100} \quad (\text{since } \frac{10}{100} = \frac{1}{10}) \\ &= 6 + \frac{3}{10} + \frac{2}{100} \quad (\text{since } \frac{10}{10} = 1) \\ &= 6.32 \end{aligned}$$

In these steps, you should be able to point our where we have used the associative and commutative properties of addition of rational numbers.

Subtraction computations with decimal fractions are done in a way similar to addition computations, as the following example illustrates.

Example 1. Subtract 4.387 from 12.125.

$$\begin{array}{r} 12.125 \\ \underline{-4.387} \\ 7.738 \end{array}$$

(We can "check" this result by noting that $7.738 + 4.387 = 12.125$.)

The quotient of two rational numbers may also be computed when decimal fractions are used to represent the numbers. First, consider the quotient $.125 \div .5$. We may express this quotient as

$$\frac{.125}{.5},$$

and we know this is the same as

$$\frac{.125}{.5} \times \frac{10}{10}. \qquad \text{(Why?)}$$

Furthermore, $\frac{.125}{.5} \times \frac{19}{10} = \frac{1.25}{5}$

Therefore, instead of working with the quotient $\frac{.125}{.5}$, we may compute the equivalent quotient $\frac{1.25}{5}$. The computation is shown below:

$$\begin{array}{r} .25 \\ 5\overline{)1.25} \\ \underline{1\,0\ } \\ 25 \\ 25 \end{array}$$

This process is justified by the following:

$$\frac{1.25}{5} = \frac{1}{5} \times 1.25 = \frac{1}{5} \times \left(\frac{1}{100} \times 125\right) = \frac{1}{100} \times \left(\frac{1}{5} \times 125\right)$$

$$= \frac{1}{100} \times 25 = .25.$$

In the preceding division problem we multiplied the given quotient $\frac{.125}{.5}$ by $\frac{10}{10}$ so that we obtained the equivalent quotient $\frac{1.25}{5}$, in which the denominator (divisor) is a whole number. If we try the same approach with the quotient

$$\frac{.0221}{.13},$$

we choose to multiply by $\frac{100}{100}$. (Do you see why?) Thus,

$$\frac{.0221}{.13} = \frac{.0221}{.13} \times \frac{100}{100}$$

$$= \frac{2.21}{13}$$

```
       .17
   13 )2.21
       1 3
         91
         91
```

$$= .17$$

Therefore, $\frac{.0221}{.13} = .17$.

Question: What is the product .17 × .13?

Often, quotients of rational numbers (expressed by decimal fractions) need be carried out only to a specified number of decimal places. Study the example below, in which the quotient has been computed correct to two decimal places (hundredths).

Example 2. What is the quotient when 253.42 is divided by 8.7?

$$\frac{253.42}{8.7} = \frac{253.42}{8.7} \times \frac{10}{10} = \frac{2534.2}{87}$$

```
        29.128
   87 )2534.200
        174
         794
         783
          11 2
           8 7
           2 50
           1 74
             760
             696
              64
```

Therefore, correct to two decimal places, the quotient is 29.13. That is,

$$\frac{253.42}{8.7} \approx 29.13$$

Questions: What is the product 29.13 × 8.7? Why is this product not equal to 253.42?

14.4 Exercises.

1. Compute the following:

(a) 2.56 + 8.94
(b) 10.487 + 35.733
(c) 42.56 − 387.29
(d) 4.5 × 2.5
(e) 2.25 × 2.25
(f) −3.5 × .4
(g) −4.85 + −6.15
(h) 21.5 − (−7.6)
(i) 55.0 − 39.8
(j) 39.8 − 55.0
(k) 4.5 × .45
(l) −8.65 − 7.15

2. Compute the following quotients.

(a) $\frac{4.08}{2.4}$ (b) $\frac{40.8}{24}$ (c) $\frac{.408}{.24}$ (d) $\frac{408}{240}$

3. Explain why all the quotients in Exercise 2 are the same.

4. Compute the following quotients, correct to two decimal places. (See Example 2).

(a) $\frac{40.8}{2.6}$
(b) 312.48 ÷ 48.4
(c) $\frac{580}{3.2}$
(d) $\frac{.05}{3.2}$
(e) $\frac{.005}{.32}$
(f) 875.42 ÷ .17

5. During one month, Mr. Sales makes the following deposits in his bank:

$42.50, $97.28, $10.12, $106,77

What is the total of these deposits?

6. At the beginning of the month, Miss Lane's bank balance was $412.65. During the month she wrote checks for the following amounts:

$5.79, $36.48, $10.20, $75.00, and $85.80.

Also, during the month, she made one deposit of $85.80. What was her bank balance at the end of the month?

7. (a) Find the quotient $\frac{3}{4} \div \frac{5}{8}$

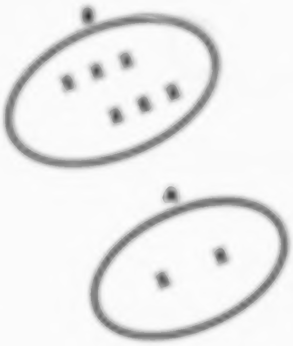
(b) Find the same quotient as in part (a) by expressing each number by a decimal fraction.

8. If the length of segment X is 3.75 inches, what is the length of segment V = (1.8)X?

9. If a certain material sells for $.45 a yard, how many yards can be bought for $5.40?

5 Ratio and Proportion.

At the right are two sets of elements, A and B. The
mber of elements in
A is 2, and the num-
of elements in set B
6. We could say that
number of elements
B is 4 more than the
mber of elements in A.
d there is another com-
n way of comparing the
izes of the two sets; this
by stating that the num-
r of elements in B is 3
mes the number of elements in A. That is, $2 \cdot 3 = 6$; or,
hat amounts to the same thing,



$$\frac{6}{2} = 3.$$

e have used the quotient $\frac{6}{2}$ to compare the sizes of
he two sets; when used in this way, a quotient is
alled a *ratio*. And the equation above may be read
s:

The ratio of 6 to 2 is 3.

Furthermore, there is another way to write $\frac{6}{2} = 3$ when
ou mean a ratio. It is as follows:

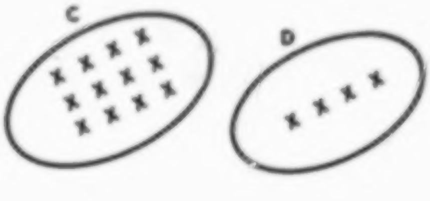
$$6:2 = 3.$$

Before looking at another example, notice that we
may say:

The ratio of B to A is 3. (even though the ratio really involves numbers)

And this means that if the number of elements in A is multiplied by 3, you get the number of elements in B.

Pictured below are two more sets, C and D, which have 12 elements and 4 elements respectively. What is the *ratio* of the number of elements in C to the number of elements in D?



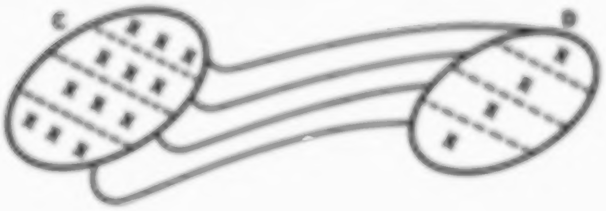
The ratio is $\frac{12}{4}$ (or 12: 4): and since $\frac{12}{4} = 3$, there are 3 times as many elements in C as in D. Or again, if the number of elements in D is multiplied by 3, the result is the number of elements in C.

Notice that in the two examples above, the ratios (quotients) are equal. That is, $\frac{6}{2} = \frac{12}{4} = 3$. This is true even though the sizes of the sets in the two examples are not the same. A sentence such as

$$\frac{6}{2} = \frac{12}{4},$$

which shows that two ratios are equal, is called a *proportion*. The sentence is sometimes written as "6:2 = 12:4." In this example, we see that $6 \cdot 4 = 2 \cdot 12$. And, in general, two ratios $\frac{a}{b}$ and $\frac{c}{d}$ are equal if $ad = bc$. Hence, the test for equal ratios is the same as the test for equivalent fractions which was given in Chapter 12.

In terms of the sets being compared, what does it mean to say that two ratios are equal? In the examples above, it means of course that in each case one set is 3 times as large as the other.



The above diagram shows for each element in D, there are 3 elements in C. Thus, the sets C and D compare (by means of a ratio) in the same way as a set having 3 elements and a set having 1 element.

Question: Can you draw a diagram like the one above which shows that for every element in A there are 3 elements in B?

Example 1. In Congress, 80 Senators voted on a certain bill, and it passed by 3:1. How many Senators voted for the bill?

This is a kind of language often used, and what it means is that the ratio of the number voting against the bill is 3:1. It does *not* mean that only 3 Senators voted for the bill, and only 1 against. As a matter of fact, in this case 60 Senators voted "yes" and 20 voted "no". Do you see why?

Example 2. Two line segments have been drawn below. Segment CD has a length of $\frac{1}{2}$ inch, and segment AB has a length of $2\frac{1}{2}$ inches. How do the two segments compare?

$$2\frac{1}{2} \div \frac{1}{2} = \frac{5}{2} \div \frac{1}{2}$$
$$= \frac{5}{2} \cdot \frac{2}{1}$$
$$= 5.$$

Thus, AB:CD = 5. The length of $\overline{AB}$ is 5 times the length of $\overline{CD}$.

Example 2 illustrates that the use of the word "ratio" is not restricted to the comparison of two whole numbers: we may also speak of the ratio of two rational numbers. In general, we say:

The ratio of a number *c* to a number *d*, $d \neq 0$, is the quotient $\frac{c}{d}$, which may also be written c:d.

Example 3. Let *g* be the number of girls in a seventh grade class, and let *b* be the number of boys. If g = 12 and b = 16, what is the ratio g:b?

$$g:b = \frac{g}{b} = \frac{12}{16} = \frac{3}{4}.$$

The two sets compare in the same way as two sets having 3 and 4 elements. For every 3 girls, there are 4 boys.

Notice also that $\frac{3}{4} \cdot 16 = 12$.

Example 4. Using the numbers from Example 3, what is the ratio b:g?

$$\frac{b}{g} = \frac{16}{12} = \frac{4}{3}. \qquad \frac{4}{3} \cdot 12 = 16.$$

From all of the examples thus far, the following generalization should be clear:

If c:d = r, then $r \cdot d = c$.

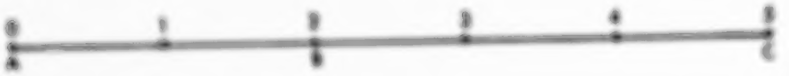
Example 5. Segment $\overline{AB}$ has a length of 24 inches, and segment $\overline{CD}$ has a length of 8 feet. What is the ratio AB:CD?
Be careful! It is tempting to say that the ratio is $\frac{24}{8} = 3$. But this is misleading, for it suggests that the length of segment CD must be multiplied by 3 to get the length of AD: but actually the length of $\overline{CD}$ is greater than that of $\overline{AB}$, since 8 feet is certainly more than 24 inches. Since the length of $\overline{CD}$ is measured in feet, we can also express the measurement of $\overline{AB}$ in feet: the length of $\overline{AB}$ is 2 feet. Then the ratio AB: CD is

$$\frac{2}{8} = \frac{1}{4}$$

The length of $\overline{AB}$ is $\frac{1}{4}$ of the length of $\overline{CD}$.

14.6 Exercises.

1. In the drawing below, two segments, $\overline{AB}$ and $\overline{AC}$, have been marked.



(a) What is the ratio of AB:AC?
(b) For what dilation D_a would the image of segment $\overline{AC}$ be segment $\overline{AB}$?
(c) What is the ratio AC:AB?
(d) For what dilation D_b would the image of segment $\overline{AB}$ be segment $\overline{AC}$?
(e) If r_1 is the ratio AB:AC, and r_2 is the ratio AC:AB, what is the product r_1r_2?

2. Find the ratio of the length of U to the length of V if:
(a) the measurement of U is 10 inches: the measurement of V is 5 inches.
(b) the measurement of U is 5 inches: the measurement of V is 10 inches.
(c) the measurement of U is 3 yards: the measurement of V is 18 inches.
(d) the measurement of U is 1 mile: the measurement of V is 2000 feet.
(e) the measurement of U is $3\frac{1}{4}$ inches: the measurement V is $1\frac{3}{4}$ inches.
(f) the measurement of U is $1\frac{3}{4}$ inches: the measurement of V is $3\frac{1}{4}$ inches.
(g) the measurement of U is $2a$ inches: the measurement of V is *a* inches. ($a \neq 0$)

3. Let *a* be the number of questions on a test. Let *b* be the number of questions a student answered correctly. Let *c* be the number of questions answered incorrectly.
If a = 20, b = 17, and c = 3, find the following:
(a) the ratio of *b* to *a*
(b) the ratio of *c* to *a*
(c) the ratio of b+c to *a*
(d) the ratio of *b* to *c*
(e) the ratio of *c* to *b*.

4. If *x* and *y* are two rational numbers such that $x:y = \frac{1}{3}$ give five possible pairs of values for *x* and *y*.

5. If *c* and *d* are two rational numbers, which number is greater if:
(a) $c:d = \frac{2}{3}$ (b) $c:d = \frac{3}{2}$ (c) c:d = 7 (d) c:d = 1 ?

If a and b are two rational numbers such that $\frac{a}{b} = \frac{3}{4}$,

(a) by what number must you multiply b to get a?

(b) by what number must you multiply a to get b?

7. Sometimes ratios are formed in which the numerator and denominator are numbers resulting from measurements involving different units. For example, on a map a ratio such as 1 inch: 3 miles means that a segment of 1 inch on the map actually represents a segment of 3 miles on the countryside. Thus we have the proportional sequences

1, 2, 3, 4, 5, . . .
3, 6, 9, 12, 15, . . .

so that a segment on the map that measures 4 inches, for example, actually represents a segment with measurement 12 miles.

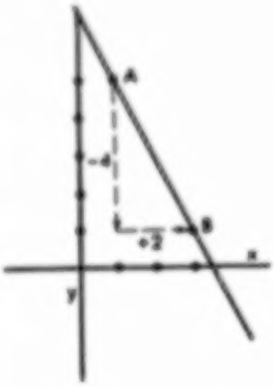
(a) On the map described above, a $6\frac{1}{2}$ inch segment represents a segment of what length?

(b) How long a segment must be drawn on the map to represent a 17 mile segment?

8. Thus far we have used only positive numbers in forming ratios. There are problems, however, in which it is sensible to use negative numbers. For example, in the drawing at the right, a line has been drawn in the plane, and two points, A and B, have been marked on the line. The coordinates of B are (3,1). Notice in "moving" from A to B, the x-coordinate increases by 2, which we indicate by +2, and the y-coordinate decreases by 4, which we indicate by −4. Now if we form the ratio

$$\frac{\text{change in y-coordinate}}{\text{change in x-coordinate}},$$

we get $\frac{-4}{+2}$, or −2. Furthermore, we say that the *slope* of the line is −2.



Using this definition of slope, complete the following activities.

(a) Mark the point (3,4), and through this point draw a line whose slope is $\frac{2}{1} = 2$.

(b) Through the point (3,4), draw a line whose slope is $\frac{-2}{1} = -2$.

(c) Mark the point (−2,5), and through this point draw a line whose slope is $\frac{-2}{3}$.

(d) Through the point (−2,5), draw a line whose slope is $\frac{2}{3}$.

(e) Through the point (0,0) draw two lines, one with slope $\frac{4}{5}$ and the other with slope $\frac{-5}{4}$. How do the two lines seem to related?

(f) Draw two lines, each with slope $\frac{1}{3}$. Draw one line through the point (0,6), and the other through the point (0,2). How do the two lines seem to be related?

14.7 Proportional Sequences.

Look at the following two sequences, S_1 and S_2, with the numbers matched as shown:

S_1: 1, 2, 3, 4, 5, 6, 7, ..., k, ...
S_2: 2, 4, 6, 8, 10, 12, 14, ..., 2k, ...

Now let us form a sequence of *ratios* by using each pair of matched numbers, the numerators taken from S_1, and the denominators from S_2. Here are the ratios we get:

$$\frac{1}{2}, \frac{2}{4}, \frac{3}{6}, \frac{4}{8}, \frac{5}{10}, \frac{6}{12}, \frac{7}{14}, \ldots, \frac{k}{2k}, \ldots$$

Notice that all of the ratios are equal. For this reason, we say that the sequences S_1 and S_2 are *proportional sequences*, and $\frac{1}{2}$ is called the *proportionality constant.*

Question: The two sequences 1, 2, 3, 4, 5, ...
and
2, 4, 6, 8, 10, ...
are *not* proportional sequences. Why not?

Returning to the sequences S_1 and S_2, we see that each of them continues without end. For instance, the number 51 is in sequence S_1: what number in S_2 matches with it?

S_1: 1, 2, 3, 4, 5, 6, ..., 51, ...
S_2: 2, 4, 6, 8, 10, 12, ..., x, ...

Although it is easy in this case to tell what number x is, we could set up the following proportion:

$$\frac{1}{2} = \frac{51}{x}.$$

Since we want the ratios to be equal, we have: $1 \cdot x = 2.51$
$x = 102.$

Therefore,

$$\frac{1}{2} = \frac{51}{102}.$$

Question: Can you show that we would have obtained the same result if we had used $\frac{2}{4}$ instead of $\frac{1}{2}$ for the proportionality constant?

Suppose we use the same two sequences, but "reverse" the order in which we consider them, like this:

S_1: 2, 4, 6, 8, 10, 12, ..., 2k, ...

S_2: 1, 2, 3, 4, 5, 6, ..., k, ...

Now, if we form ratios as we did before, selecting the numerators from S_1 and the denominators from S_2, we get:

$$\frac{2}{1}, \frac{4}{2}, \frac{6}{3}, \frac{8}{4}, \frac{10}{5}, \frac{12}{6}, \ldots, \frac{2k}{k}, \ldots$$

Do you see that the sequences are still *proportional*? Now, however, the proportionality constant is $\frac{2}{1}$. And if we were to solve the problem we solved earlier, the proportion would look like this:

$$\frac{2}{1} = \frac{x}{51}.$$

Do you see that we would again find x to be 102?

Question: The two proportionality constants we found by considering the sequences in two different orders were $\frac{1}{2}$ and $\frac{2}{1}$. How are they related?

Consider next the two sequences below.

S_1: 3, 6, 9, 12, 15, 18, ...

S_2: 4, 8, 12, 16, 20, 24, ...

Do you see that the sequences are proportional, and that, considering the sequences as we have them, the proportionality constant is $\frac{3}{4}$? Suppose we ask: What number in S_2 corresponds to the number 10 in S_1? The question may seem to be an odd one, since the number 10 is not in the sequence S_1. What we are really asking is this: If 10 is "inserted" in S_1, what number must be "inserted" in S_2, so that the resulting sequences are proportional, with proportionality constant still $\frac{3}{4}$? The new sequences will look like this:

3, 6, 9, 10, 12, 15, ...

4, 8, 12, x, 16, 20, ...

And we find x by solving the following proportion:

$$\frac{3}{4} = \frac{10}{x}.$$

Then we have:

$$3 \cdot x = 4 \cdot 10$$

$$3 \cdot x = 40$$

$$x = \frac{40}{3}$$

Of course, $\frac{40}{3} = \frac{39}{3} + \frac{1}{3} = 13 + \frac{1}{3}$. So we can also say that x is $13\frac{1}{3}$.

Example 1. A picture has measurements of 7 inches ("length") and 3 inches ("width"). If the picture is to be enlarged so that the new length is 10 inches, what must the new width be?

The numbers 7 and 3 suggest the following proportional sequences:

S_1: 7, 14, 21, 28, 35, ...

S_2: 3, 6, 9, 12, 15, ...

where the measures of length are taken from S_1 and the measures of width from S_2. We want the ratio of length to width to be 7:3. From the sequences, we can see that if the length is made to be 14 inches, then the width must be 6 inches: if the length is made to be 21 inches, then the width must be 9 inches: etc. However, in our problem the length is to be 10 inches. The number 10 is not in S_1 as we have it. So we can form the sequences

7, 10, 14, 21, 28, 35, ...

3, x, 6, 9, 12, 15, ...

and find what number x must be so that the sequences are proportional with proportionality constant $\frac{7}{3}$. We solve the problem as follows:

$$\frac{7}{3} = \frac{10}{x}$$

$$7 \cdot x = 3 \cdot 10$$

$$7 \cdot x = 30$$

$$x = \frac{30}{7}.$$

Therefore, the width of the enlarged picture must be $4\frac{2}{7}$ inches.

Example 2. Solve the proportion

$$\frac{3}{8} = \frac{x}{28}.$$

We find the number x which will make the following sequences proportional:

S_1: 3, 6, 9, x, 12, ...

S_2: 8, 16, 24, 28, 32, ...

We solve the proportion as follows:

$$3 \cdot 28 = 8 \cdot x$$
$$8 \cdot x = 84$$
$$x = 10\frac{1}{2}$$

In other words, $\frac{3}{8} = \frac{10\frac{1}{2}}{28}$

.8 Exercises.

1. Using whole numbers only,
   (a) Write two proportional sequences with proportionality constant $\frac{5}{6}$.
   (b) Write two proportional sequences with proportionality constant $\frac{1}{4}$.
   (c) Write two proportional sequences with proportionality constant .5.
2. In each of the following, find what number x must be so that the two sequences are proportional.

   (a) S_1: 2, 4, 6, 8, 10, ...
   S_2: 9, 18, x, 36, 45, ...

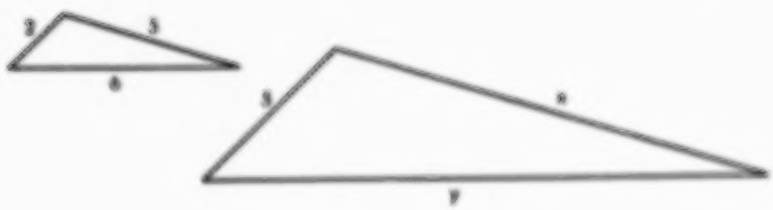
   (b) S_1: 7, 14, 21, x, 35, ...
   S_2: 3, 6, 9, 12, 15, ...

   (c) S_1: 3, 6, 9, 10, 12, ...
   S_2: 5, 10, 15, x, 20, ...
3. Solve the following proportions.

   (a) $\frac{5}{2} = \frac{15}{x}$ (d) $\frac{100}{21} = \frac{7}{x}$ (g) $\frac{\frac{1}{2}}{3} = \frac{x}{12}$

   (b) $\frac{5}{2} = \frac{12}{x}$ (e) $\frac{2}{1} = \frac{9}{x}$ (h) $5:3 = x:15$

   (c) $\frac{3}{7} = \frac{3}{x}$ (f) $\frac{1}{2} = \frac{9}{x}$ (i) $\frac{x}{10} = \frac{a}{2a} (a \neq 0)$
4. The ratio of number of boys to number of girls is the same in two different seventh grade classes. In one class, there are 12 boys and 16 girls. In the second class, there are 15 boys. What is the *total number of students* in the second class?
5. On a certain map there are two segments drawn, one 7 inches long and the second 10 inches long. If the map is enlarged so that the first segment measures 25 inches, how long will the second segment be in the enlargment?
6. Two triangles are drawn below. The triangles are similar, which means that the ratios of corresponding sides are all the same. All of the sides in one triangle have their lengths indicated in the figure. In the other triangle, the length of only one side has been marked. Find the lengths, x and y of the other sides.



14.9 Meaning of Per Cent.

Consider the sequences below:

S_1: 2, 4, 6, 8, 10, ..., 40, ..., 2k, ...

S_2: 5, 10, 15, 20, 25, ..., 100, ..., 5k, ...

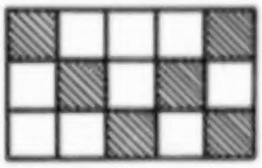
It is easy to see that the proportionality constant is the ratio $\frac{2}{5}$. The ratio $\frac{40}{100}$, which arises from these sequences, is especially important in many applications of mathematics because the denominator is 100. The ratio $\frac{40}{100}$ may be written as

40% (read "forty per cent").

We can also use a decimal number in referring to the ratio, as follows:

$$\frac{40}{100} = .40 = 40\%$$

Example 1. In the picture below, there are 15 square regions, and 6 of them have been shaded. What *per cent* of the squares are shaded?



The number of shaded squares is 6: the total number of squares is 15. So the ratio of the number of shaded squares to the total number of squares is

$$\frac{6}{15}.$$

And we can say that $\frac{6}{15}$ of the squares are shaded. However, from the discussion above, we know that

$$\frac{6}{15} = 40\%. \qquad \text{(Why?)}$$

Therefore, 40% of the squares are shaded.

Example 2. Express $\frac{3}{8}$ as a per cent.

We may express this problem in terms of the following proportional sequences:

S_1: 3, 6, 9, 12, 15, ..., x, ..., 3k, ...

S_2: 8, 16, 24, 32, 40, ..., 100, ..., 8k, ...

Then we solve the proportion

$$\frac{3}{8} = \frac{x}{100}.$$

$$8 \cdot x = 3 \quad 100$$

$$8 \cdot x = 300$$

$$x = \frac{300}{8} = 37\frac{1}{2}.$$

Therefore, $\frac{3}{8} = \frac{37\frac{1}{2}}{100} = 37\frac{1}{2}\%$. And we say that $37\frac{1}{2}\%$ is the per cent equivalent of $\frac{3}{8}$.

Example 3. Find the per cent equivalent of $\frac{6}{5}$.

We use the proportion $\frac{6}{5} = \frac{x}{100}$. (That is, we want a ratio with denominator 100 that is equal to the ratio $\frac{6}{5}$.)

$$6 \cdot 100 = 5 \cdot x$$

$$5 \cdot x = 600$$

$$x = \frac{600}{5} = 120.$$

Therefore, $\frac{6}{5} = 120\%$.

Questions: In a ratio $\frac{a}{b}$, how must a and b be related so that the per cent equivalent of the ratio is greater than 100%?
less than 100%?
equal to 100%?

Example 4. What is the per cent equivalent of 3.5?

$$3.5 = 3\frac{5}{10} = 3\frac{50}{100} = \frac{350}{100} = 350\%.$$

Example 5. Express $\frac{1}{2}\%$ as the ratio of two whole numbers.

$\frac{1}{2}\% = \frac{\frac{1}{2}}{100}$. This is a ratio, but it is not a ratio of whole numbers. However, we know that

$$\frac{\frac{1}{2}}{100} = \frac{\frac{1}{2}\cdot 2}{100\cdot 2} = \frac{1}{200}$$

(Why?)

Therefore, $\frac{1}{2}\% = \frac{1}{200}$.

Question: Which is greater, $\frac{1}{2}$ or $\frac{1}{2}\%$?

Having looked at a number of particular cases, we might consider the general problem of finding the per cent equivalent of a ratio. Let $\frac{a}{b}$ be any ratio (of course, $b \neq 0$). Then to say that $\frac{a}{b} = x\%$ is to say $\frac{a}{b} = \frac{x}{100}$. Then we have:

$$b \cdot x = 100 \cdot a$$

$$x = \frac{100a}{b}$$

Otherwise stated, $\frac{a}{b} = \frac{100a}{b}\%$.

14.10 Exercises.

1. (a) 50% is the percent equivalent of $\frac{1}{2}$. Write four other ratios for which 50% is the percent equivalent.
   (b) Write five different ratios for which 25% is the percent equivalent.
   (c) Write five different ratios for which 150% is the percent equivalent.
   (d) Write five different ratios for which 100% is the percent equivalent.
   (e) Write five different ratios for which 200% is the percent equivalent.
2. The questions in this exercise refer to the figure below.

A		C		B
	A	C		B
C	C	A		
	C		A	

   (a) What percent of the squares have been marked "A"?
   (b) What percent of the squares have been marked "B"?
   (c) What percent of the squares have been marked "C"?
   (d) What percent of the squares have no mark?
   (e) What is the *sum* of the percents in questions (a), (b), (c), and (d)?
3. Give the percent equivalent of each of the following:
   (a) .5 (b) .50 (c) .25 (d) 2.5 (e) 1.5
   (f) 1.25 (g) .17 (h) 1.17
4. In the table below, each ratio is to be expressed in the form $\frac{a}{b}$, as a decimal fraction, and as a percent. The first row has been filled in correctly. Fill in all the blanks in the remainder of the table.

Ratio $\frac{a}{b}$	Decimal Fraction	Per Cent
1/2	.50	50%
1/4		
	.75	
		20%
	.60	
	.20	
1/8		
		87 1/2%
4/5		
	.375	
		40%
1/10		
		90%
1/1		
	.70	
	.05	
3/10		
		1%

5. As you recall from Section 12.23, some ratios such as $\frac{1}{3}$ cannot be expressed as terminating decimals, but can be approximated to any desired number of decimal places. How can such a ratio as $\frac{1}{3}$ be expresses as a per cent? The question is answered in the same way that all other problems concerning percent equivalents have been answered. Study the steps below:

$$\frac{1}{3} = \frac{x}{100}$$
$$3 \cdot x = 1 \cdot 100$$
$$3 \cdot x = 100$$
$$x = \frac{100}{3} = 33\tfrac{1}{3}$$

Therefore, the ratio $\frac{1}{3}$ may be expressed as $33\frac{1}{3}$%.

Give the percent equivalent of the following ratios:

(a) $\frac{2}{3}$ (b) $\frac{1}{6}$ (c) $\frac{5}{6}$ (d) $\frac{1}{12}$

14.11 Solving Problems with Per Cents

It is common to see advertisements with statements such as

SALE: 15% OFF ON ALL ITEMS!

Suppose that an item that normally sells for \$25.00 is included in the sale advertised above. What should the sale price be? We know that a certain amount should be *subtracted* from the normal price of \$25.00, but how much? According to the advertisement, 15% of 25.00 should be subtracted. So the problem is that of finding 15% of 25. Since 15% means $\frac{15}{100}$, we may work with the following proportional sequences:

$$S_1:\ 3,\ x,\ 6,\ 9,\ 12,\ 15, \ldots, 3k, \ldots$$
$$S_2:\ 20,\ 25,\ 40,\ 60,\ 80,\ 100, \ldots, 20k, \ldots$$

Do you see that the proportionality constant is $\frac{3}{20}$ or $\frac{15}{100}$? The question is: What value of x will make the ratio $\frac{x}{25}$ equal to the ratio $\frac{15}{100}$? We solve the following proportion:

$$\frac{x}{25} = \frac{15}{100}$$
$$100 \cdot x = 25 \cdot 15$$
$$100 \cdot x = 375$$
$$x = \frac{375}{100} = 3.75$$

Therefore, the amount to be subtracted is \$3.75 (which is 15% of \$25.00). And since \$25.00 − \$3.75 = \$21.25, the item should sell for \$21.25 during the sale.

In the following examples, we solve some other problems, all by use of percents.

Example 1. On a test having 20 questions, a student answered 16 of them correctly. What per cent of the questions did he answer correctly? That is, what should his per cent score be?

The ratio of the number of questions answered correctly to the total number of questions is $\frac{16}{20}$. So, the student answered $\frac{16}{20}$ of the questions correctly. But we can also say that he answered $\frac{4}{5}$ of the questions correctly. (Why?) Finally, since we already know that $\frac{4}{5}$ = 80%, we can say that he answered 80% *of the questions* correctly.

Example 2. On the same test of 20 questions, another student missed 3. What is his per cent score? Since the student missed 3, he answered 17 correctly. The ratio

$$\frac{\text{number correct}}{\text{total number}} \text{ is } \frac{17}{20}.$$

We want an equal ratio in which the denominator is 100.

If $\frac{17}{20} = \frac{x}{100}$, then $20 \cdot x = 1700$ or

$x = \frac{1700}{20} = 85.$

Hence, the student's per cent score is 85%.

Example 3. In a certain election, 70% of a town's registered voters actually voted. If 3,780 people voted, how many registered voters are in the town?

Certainly we know that $70\% = \frac{70}{100}$.

And we know that this is the ratio

$$\frac{\text{number who voted}}{\text{number of registered voters}}.$$

Since we know the number who voted 3780, we have the following proportion:

$$\frac{70}{100} = \frac{3780}{x}$$

$$70 \cdot x = 3780 \cdot 100$$

$$70 \cdot x = 378{,}000$$

$$x = \frac{378000}{70} = 5400.$$

Therefore, there are 5400 registered voters in the town. This could be checked by showing that 70% of 5400 is 3780.

Example 4. A major league ball player has been at bat 82 times, and collected 26 hits. What is his "batting average"?

The ratio $\frac{\text{number of hits}}{\text{number of times at bat}}$ is

$\frac{26}{82}$ or $\frac{13}{41}$.

We find the per cent equivalent from the following proportion:

$$\frac{13}{41} = \frac{x}{100}$$

$$41 \cdot x = 13 \cdot 100$$

$$41 \cdot x = 1300$$

$$x = \frac{1300}{41}$$

As usual, we interpret $\frac{1300}{41}$ to mean $1300 \div 41$. The steps in carrying out this division are shown at the right. Notice that the quotient is *approximately* 31.7. Therefore, we can write

$$\frac{13}{41} = \frac{31.7}{100}$$

```
      31.7
 41)1300.0
    123
     70
     41
     29 0
     28 0
      1 0
```

So the batting average is approximately 31.7%. If you are a baseball fan, you probably know that this average is more likely to be listed as .317.

Example 5. What is $\frac{3}{4}\%$ of 280?

Important! The answer *is not 210*. (Don't confuse $\frac{3}{4}\%$ with $\frac{3}{4}$) $\frac{3}{4}\%$ is equal to

$$\frac{\frac{3}{4}}{100} = \frac{\frac{3}{4} \cdot 4}{100 \cdot 4} = \frac{3}{400}.$$

So we are really finding $\frac{3}{400}$ of 280. We may find it from the proportion

$$\frac{3}{400} = \frac{x}{100};$$

or we may solve the problem as we did in Section 12.8 :

$$\frac{3}{400} \text{ of } 280 = \frac{3}{400} \cdot 280 = \frac{840}{400} = 2.10.$$

Therefore,

$$\frac{3}{4}\% \text{ of } 280 \text{ is } 2.10.$$

All of the common types of per cent problems may be solved by using proportions. But if you understand the meaning of per cent, you can often solve problems very quickly without use of a formal proportion. For instance, look again at Example 5: What is $\frac{3}{4}\%$ of 280? $\frac{3}{4}\%$ of a number is $\frac{3}{4}$ of 1% of the number. And 1% of 280 is 2.80. So the result can be found by taking $\frac{3}{4}$ of 2.80, which is 2.10. Feel free to use such methods in the following exercises; but if you are in doubt, you can always use a proportion.

14.12 Exercises

1. Find the following:

   (a) 1% of 500; 5% of 500; $\frac{1}{2}\%$ of 500; $1\frac{1}{2}\%$ of 500; $\frac{1}{2}$ of 500.

(b) 1% of 150; 10% of 150; $\frac{1}{3}\%$ of 150; $1\frac{1}{3}\%$ of 150.

(c) 1% of 24; 28% of 24; $\frac{3}{4}\%$ of 24; $1\frac{3}{4}\%$ of 24; $\frac{3}{4}$ of 24.

(d) 1% of 8000; .5% of 8000; 1.5% of 8000; 4.5% of 8000; .5 of 8000.

(e) 1% of 50; 100% of 50; 200% of 50; 240% of 50,

(f) 1% of 92; 100% of 92; 300% of 92; 350% of 92.

In a high school with 2600 students, 35% of the students are freshmen. How many students are freshmen?

In the same high school, there are 390 seniors. What per cent of the school's students are seniors?

Suppose the town of Elmwood has a population of 4000, and the town of Springfield has a population of 6000. Complete the following statements.

(a) The ratio of Elmwood's population to Springfield's population is

(b) Elmwood's population is ____% of Springfield's population.

(c) The ratio of Springfield's population to Elmwood's population is .

(d) Springfield's population is ___% of Elmwood's population.

Complete the statements in the following two columns in the same way the first statement in each column has been completed.

$20 = \frac{20}{40} \cdot 40$	20 is 50% of 40.
$40 =$ ___ $\cdot 20$	40 is ___% of 20.
$20 =$ ___ $\cdot 25$	20 is ___% of 25.
$25 =$ ___ $\cdot 20$	25 is ___% of 20.
$500 =$ ___ $\cdot 400$	500 is ___% of 400.
$400 =$ ___ $\cdot 500$	400 is ___% of 500.
$8 =$ ___ $\cdot 80$	8 is ___% of 80.
$80 =$ ___ $\cdot 8$	80 is ___% of 8.
$16 =$ ___ $\cdot 80$	16 is ___% of 80.
$80 =$ ___ $\cdot 16$	80 is ___% of 16.
$4.2 =$ ___ $\cdot 42$	4.2 is ___% of 42.
$42 =$ ___ $\cdot 4.2$	42 is ___% of 4.2.
$1.8 =$ ___ $\cdot 180$	1.8 is ___% of 180.
$180 =$ ___ $\cdot 1.8$	180 is ___% of 1.8.

6. In a basketball game, a high school team scored 80 points.

(a) If David scored 18 of these points, what per cent of the team's points did he score?

(b) Bill made $27\frac{1}{2}\%$ of the team's points. How many points did he score?

(c) The number of points David scored is *what per cent* of the number of points Bill scored?

7. In another game, David made 40% of the team's points. If he made 22 points, how many points did the entire team make?

8. (a) 22 is 40% of what number?

(b) 80 is 50% of what number?

(c) 12 is 35% of what number?

(d) 60 is 150% of what number?

(e) 7 is 1% of what number?

(f) 42 is $\frac{1}{2}\%$ of what number?

9. In a certain state, there is a 4% sales tax. How much sales tax must be paid on purchases of the following amounts:

(a) \$40.00 (d) \$3.25 (g) \$3500.00

(b) \$15.00 (e) \$1.00 (h) \$3499.00

(c) \$12.50 (f) \$10.00 (i) \$9.99

10. Suppose a bank pays $4\frac{1}{2}\%$ interest per year on savings deposits.

(a) How much interest should a deposit of \$2000 earn in one year?

(b) How much interest should a deposit of \$2000 earn in two years?

11. If the bank in problem 10 pays interest every six months, it will pay only half as much, since 6 months is $\frac{1}{2}$ of a year. (It is the *annual* interest rate which is $4\frac{1}{2}\%$.)

(a) How much will \$1000 earn for six months?

(b) How much will \$2500 earn for six months?

(c) How much will \$2000 earn for three months?

(Hint: 3 months is $\frac{1}{4}$ of a year.)

From Exercises 10 and 11, we see that simple interest can be computed from the formula

$$i = p \cdot r \cdot t ,$$

where i is the interest, p is the amount of money deposited, r is the rate of annual interest, and t is the time in years.

12. Compute the interest for:

(a) \$500 at 4% for 1 year

(b) \$500 at 4% for 6 months

(c) \$500 at 4% for 3 months

(d) \$1200 at $4\frac{1}{4}\%$ for 1 year

(e) \$1200 at $4\frac{1}{4}\%$ for 6 months

(f) \$1200 at $4\frac{1}{4}\%$ for 3 months

(g) \$1500 at $5\frac{1}{2}\%$ for 2 years

(h) \$1500 at $5\frac{1}{2}\%$ for 1½ years

(i) \$750 at 4.2% for 1 year

(j) \$750 at 4.2% for 6 months.

13. Mr. Smith has kept a deposit of \$1500 in a bank for one year, and the bank pays him \$37.50 interest. What annual rate of interest is the bank paying?

14. Complete the following sentences:

(a) $33\frac{1}{3}\%$ of 3900 is_____.

(b) 20 is_____% of 30.

(c) 30 is_____% of 20.

(d) 20 is 18% of_____.

(e) 20 is 40% of_____.

(f) 108 is 40% of____.

(g) $2\frac{3}{4}\%$ of 160 is____.

(h) 2.75% of 160 is___.

(i) 18 is $66\frac{2}{3}\%$ of_____.

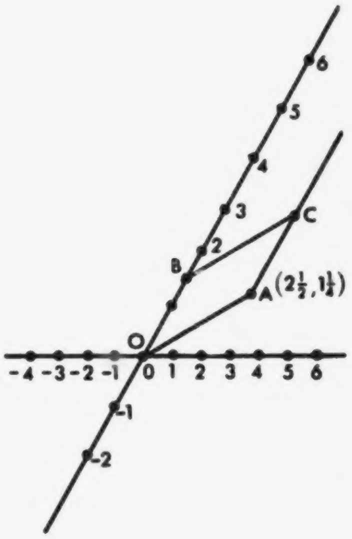
(j) $16\frac{2}{3}\%$ of 66 is_____.

(k) 30 is_____% of 36.

14.13 Translations and Groups.

In preceding chapters we studied translations of a set of points on a line onto itself; of a set of points on one of two parallel lines onto a set of points on the other; of a set of lattice points in a plane onto itself. In this section we extend translations so that they may have as a domain the set of points in a plane whose coordinates, in a coordinate system, are rational numbers.

Consider the translation, call it t, that maps 0 (o,o) onto A $(2\frac{1}{2}, 1\frac{1}{4})$. What is the image of B $(0, 1\frac{1}{2})$ under t? Name it C. What kind of figure is OACB? Why? The coordinate rule of t is $(x,y) \longrightarrow (x+2\frac{1}{2}, y+1\frac{1}{4})$. Is t a one-to-one onto mapping? Why? Does t have an inverse? Let us name it t^{-1}. The −1 denotes an inverse mapping, so t^{-1} is read "the inverse of t" or simply, "t inverse." In t^{-1} what is the image of A? of C? of 0? The rule for t^{-1} is: $(x,y) \longrightarrow (x-2½,4-1¼)$



Do you think that every translation of the set of points in a plane with rational coordinates has an inverse? If a translation has rule $(x,y) \longrightarrow (x+a, y+b)$ where x, y, a, b are rational numbers, what is the rule for the inverse of this translation?

Now consider translation t' that maps (x,y) onto $(x+3\frac{1}{4}, y-\frac{3}{4})$. Under t', what is the image of A $(2\frac{1}{2}, 1\frac{1}{4})$? Is there a single translation that maps 0 onto this image? What is its rule? Thus, there is a translation which is the composite of t' with t', and, as you recall, we denote it $t' \circ t$ (t first followed by t').

In particular, what is the composite of t with its inverse t^{-1}? Then it would seem that among the translations is the identity translation.

In summary, if t: $(x,y) \longrightarrow (x+a, y+b)$
then t^{-1}: $(x,y) \longrightarrow (x-a, y-b)$
If t^{-1}: $(x,y) \longrightarrow (x+c, y+d)$
then $t' \circ t$ $(x+a+c, y+b+d)$

You have probably suspected that the set of translation we have been discussing, together with composition, have the properties of a group. Indeed they do, and you are asked to further investigate this question in the following set of exercises.

14.14 Exercises.

Assume that all translations in the exercises have for their domain (and range), the set of all points in a plane with rational coordinates in some coordinate system, unless otherwise specified.

1. Is the composition of two translations an operation? Why?
2. Let T represent the set of all translations. List the properties that should be proved for (T,°) that will support the claim that (T,°) is a group.
3. Prove that every translation has an inverse in (T,°).
4. Prove that (T,°) contains an identity translation.
5. Prove that (T,°) has the associative property.
6. Prove that (T,°) is a commutative group.
7. Let translation $\underline{t}$ map (x,y) onto $(x+\frac{1}{3}, y-2\frac{1}{4})$.

   Find the rule for each of the following:

   (a) $\underline{t} \circ \underline{t}$

   (b) $\underline{t} \circ \underline{t} \circ \underline{t}$

   (c) $\underline{t} \circ \underline{t} \circ \underline{t} \circ \underline{t}$

   (d) If $\underline{t}$ is denoted t^1, $t \circ t$ is denoted t^2, $\underline{t} \circ \underline{t} \circ \underline{t}$ is denoted t^3, and so on, does the set $\{\underline{t}^1, \underline{t}^2, \underline{t}^3, \underline{t}^4 \ldots\}$ with ° form a group? If it does not, explain in what respect it is deficient.
8, Using the data in Exercise 7 find the rule for each of the following:

   (a) $\underline{t}^{-1}$

   (b) $\underline{t}^{-1} \circ \underline{t}^{-1}$ (denoted $\underline{t}^{-2}$)

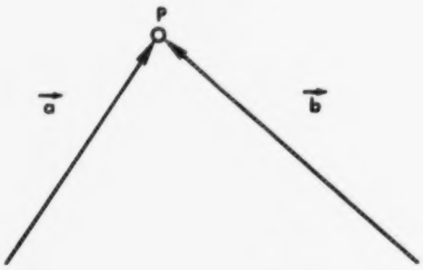
   (c) $\underline{t}^{-1} \circ \underline{t}^{-1} \circ \underline{t}^{-1}$ (denoted $\underline{t}^{-3}$)

   (d) Does the set $\{\underline{t}^{-1}, \underline{t}^{-2}, \underline{t}^{-3} \ldots\}$ with ° form a group? If not, in what respect is it deficient?
9. Does the set $\{\ldots \underline{t}^{-3}, \underline{t}^{-2}, \underline{t}^{-1}, \underline{T}, \underline{t}, \underline{t}^2\, \underline{t}^3, \ldots\}$ with ° form a group, where $\underline{T}$ is the identity transformation? If not, in what respect is it deficient?
10. Show that all translations having rules of the form $(x,y) \longrightarrow (x+pa, y+qb)$, where a and b are fixed rational numbers, and p and q are integers, form a group with °. (Difficult).

14.15 Applications of Translations.

As you might expect, translations have been studied by mathematicians because they are quite useful in solving certain types of problems. In this section we examine two of these types, both found in science. One type of problem introduces forces and the other velocities.

We shall first examine a problem involving forces. Let P, in the diagram below, represent a billiard cue ball which is about to be struck by two billiard cues at the same time. We want to know how the combined effect may be achieved with a single billiard cue.



In considering the effect of each cue we must know both the magnitude and the direction of the force which is applied to the ball by the cue. We represent the forces (not the cues) in the diagram by the line segments a and b, together with an arrow at one end of each segment. The length of each segment represents the magnitude of the force (in our diagram one inch represents a magnitude of 5 pounds). The line in which the segment lies, together with its arrow, indicates the direction of the force. Thus, one force is represented by line segment a in the direction of P. We denote this force by a. The other force is represented by line segment b in the direction of P. We denote this force by b. Since the length of of a is one inch, a has a magnitude of 5 pounds. Line segment b is 2 inches long so that the magnitude of b is 10 pounds.

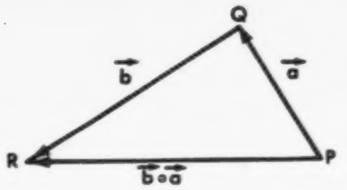
We see, then, that a force is determined by a magnitude and a direction. A translation is determined in the same way. For this reason we can expect to be able to use translations to solve our problem. Our expectations are enforced by the report of scientists that "adding" forces can be done by composing translations.

Now let us "add" the two forces $\vec{a}$ and $\vec{b}$ described above. To do this we think of P as a point and $\vec{a}$ and $\vec{b}$ as translations. Then we see, in the diagram at the right,

that $\vec{a}: P \longrightarrow Q$

$\vec{b}: Q \longrightarrow R$

Hence $\vec{b} \circ \vec{a}: P \longrightarrow R$



$\vec{b} \circ \vec{a}$ is the translation that corresponds to the "sum" of the forces. That is, the effect of $\vec{a}$ and $\vec{b}$ together will be to exert a force with a magnitude represented by PR in the line of $\overline{PR}$ and in the direction of R. This force is called the resultant of forces $\vec{a}$ and $\vec{b}$. Going back to our original problem, we see that to achieve the same effect with a single cue the cue ball would have to be struck with a force of $11\frac{1}{4}$ pounds.

Also, the cue would be sighted along $\overleftrightarrow{PR}$ in the direction from P to R.

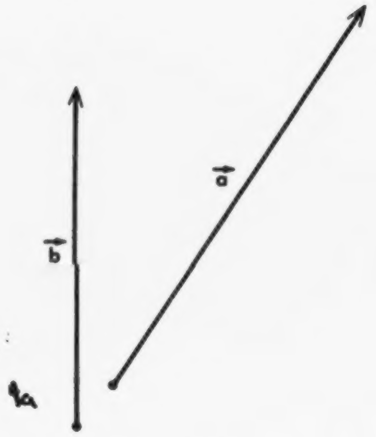
Question: Does $a \circ b = b \circ a$? Why or why not?

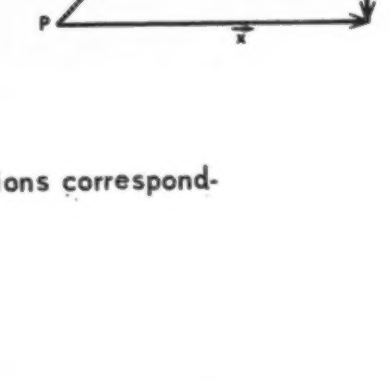
The second application of translations is to problems involving velocity. Our problem will then be to "add" velocities in same sense that we "added" forces. We can reinterpret our problem of "adding" forces $\vec{a}$ and $\vec{b}$ by thinking of them as velocities. Then $\vec{a}$ can represent a speed of 5 miles per hour in the direction indicated in the diagram and $\vec{b}$ can represent a speed of 10 miles per hour in the direction indicated in the diagram. Here again the lengths of $\vec{a}$ and $\vec{b}$ represent the magnitudes (speeds in miles per hour) of the velocity, and the line of the segment, with its arrow, represents the direction. Here we might be solving a problem such as the following:

> A toy boat is propelled by its engine with velocity $\vec{b}$. A wind is blowing with velocity $\vec{a}$. In what direction, and with what speed, does the boat actually move? (That is, with what velocity does the boat move?)

The answer is found in exactly the same manner as "adding" forces. The answer for this problem then, it: the boat moves at the rate of $11\frac{1}{4}$ miles per hour in the direction of $\overline{PR}$ as indicated by its arrow.

We end this section with another example. Suppose a boat moves in the direction of $\vec{a}$ (shown at the right) with a speed of 20 miles per hour, but its propeller and engine operate to make it move in the direction of $\vec{b}$ (shown at the right) with a speed of 15 miles per hour. The difference is due to the wind. In what direction is the wind blowing and with what speed? Note that $\vec{a}$ is 2 inches long and $\vec{b}$ is $1\frac{1}{2}$ inches long. What then is the scale in the drawing?





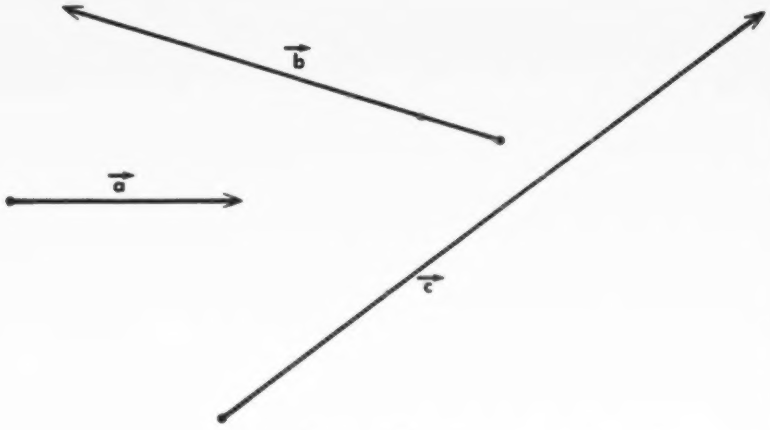
To solve this problem think of $\vec{a}$ and $\vec{b}$ as the translations corresponding to the given velocities and $\vec{x}$ as the translation corresponding to the velocity of the wind. Since $\vec{a}$ is the composite of $\vec{b}$ with $\vec{x}$ we have: $\vec{b} \circ \vec{x} = \vec{a}$. We solve for $\vec{x}$ and find $\vec{x} = \vec{b}^{-1} \circ \vec{a}$. This guides us in solving the problem. Study the diagram above and be able to explain how it was made. In looking at the diagram, start at P. How long is segment x? What is the speed of the wind?

14.16 Exercises.

1. The propeller and engines of a ship are set to propell it on an easterly course, at the speed of 20 miles per hour. The wind is moving towards the north (coming from the south) at the speed of 10 miles an hour. Make a diagram of the actual course, i.e. the velocity of the ship. Using ruler and protractor, find the actual speed and find what angle the course makes with the line pointing to the north. (Use the scale: 1 inch = 10 miles).

2. Answer the same questions asked in Exercise 1 for each of the following cases.
   (a) intended course of ship is northeast, speed of 15 miles per hour, the wind comes from the west at 30 miles per hour. (Use the scale: 1 inch = 10 miles)
   (b) intended course is northwest, speed of 18 miles per hour; the wind comes from the southwest, speed of 24 miles per hour. (Use the scale: 1 inch = 6 miles).
   (c) The ship's intended course is southeast, speed of 15 miles per hour; the wind comes from the northwest, speed of 5 miles per hour. (Do you need a diagram for this problem?)

In Exercise 3 use the segments shown below to represent forces. The scale we used to draw them is 1 inch = 10 pounds.



3. Suppose forces $\vec{a}$ and $\vec{b}$ are applied to an object. Use a diagram to find the resultant and compute the magnitude (number of pounds) in the resultant force.
4. Answer the questions in Exercise 3 for each of the following cases.

(a) forces $\vec{a}$ and $\vec{c}$ are applied together
(b) forces $\vec{b}$ and $\vec{c}$ are applied together.
(c) $\vec{a}$, $\vec{b}$, and $\vec{c}$ are applied together.

5. Suppose force $\vec{a}$ is applied and $\vec{c}$ is the resultant. Find the force $\vec{x}$ that was applied together with $\vec{a}$, and compute its magnitude.
6. Suppose force $\vec{b}$ is applied and $\vec{c}$ is the resultant. Find the force $\vec{x}$ that was applied together with $\vec{b}$, and compute its magnitude.
7. Suppose $\vec{c}$ is applied and $\vec{a}$ is the resultant. Find the force $\vec{x}$ that was applied together with $\vec{c}$ and compute its magnitude.
8. Suppose two forces are applied and the resultant leaves the object in its original position. What must have been true of the two forces? (two possible answers)

14.17 Summary.

1. If _x_ is any rational number, then D_x is a dilation which maps each point into a point x times as far from the origin. If _x_ is a negative number, the point is reflected in the origin.
2. Decimal fractions may be used in finding sums, differences, products, and quotients of rational numbers.
3. Two sets may be compared by means of a ratio. The ratio of a number _x_ to a number _y_ is the quotient $\frac{x}{y}$, also written as x:y. (It is understood that $y \neq 0$.)

   If $\frac{x}{y} = r$, then $x = r \cdot y$, and $y = \frac{x}{r}$.
4. If two sequences

   S_1: a_1, a_2, a_3, a_4, ..., a_k, ...

   S_2: b_1, b_2, b_3, b_4, ..., b_k, ...

   are related so that
   $$\frac{a_1}{b_1} = \frac{a_2}{b_2} = \ldots = \frac{a_k}{b_k} = r,$$
   then the sequences are said to be proportional sequences, and r is called the proportionality constant.

   An equation such as $\frac{a_1}{b_1} = \frac{a_2}{b_2}$ is called a proportion.
5. The ratio $\frac{a}{100}$ is also written as "a%" and read "a per cent."

   Every ratio can be expressed in the form $\frac{a}{b}$, where _a_ and _b_ are integers, or as a decimal fraction, or as a per cent.

   Many mathematics problems occurring in everyday life are expressed in the language of per cents.
6. If T is the set of all translations of form t: (x,y) (x+a,y+b), where _a_ and _b_ are rational numbers; and if ° is composition of translations, then (T,°) is a commutative group.

14.18 Review Exercises.

1. (a) What is $\frac{2}{9}$ of 18?

   (b) What is 15% of 200?

   (c) What is .35 of 650?
2. If 12% tax must be paid on $3500, how much tax must be paid?
3. During a sale, a store reduces all prices by 20%. What is the sale price of a television set which normally sells for $220.00?
4. In a school, 35 of the 225 boys go out for basketball, What per cent of the boys in the school go out for basketball?
5. 4% of the girls in the school are cheerleaders, and there are 8 girl cheerleaders. How many girls are there in the school?
6. A bank pays interest at an annual rate of $4\frac{3}{4}$%. How much will $4000 earn during a 6-month period?
7. Compute the following:

   (a) $8.875 + 44.327$

   (b) $102.54 - 87.39$

   (c) $21.8 - 39.3$

   (d) $(2.3) \times (4.3 \times 7.5)$

   (e) 5.6×8.75

   (f) $\frac{6.138}{4.65}$

   (g) $\frac{6.138}{1.32}$.
8. In a certain city there are 4200 Democrats and 3600 Republicans. What is the ratio of Democrats to Republicans? (Express the answer as an irreducible fraction.)

   Then fill in the following blanks so that a true statement results:

   For every ____Republicans, ther are ____Democrats.
9. In a student council, there are 24 members. With all members voting, Jim won the presidency by a 3:1 vote. How many voted for Jim?
10. Solve the following proportions:

    (a) $\frac{5}{3} = \frac{35}{x}$ (b) $\frac{2}{7} = \frac{9}{x}$ (c) $\frac{2}{7} = \frac{x}{9}$ (d) $\frac{9}{x} = \frac{x}{4}$.
11. Write the coordinates of the image of each of the following points under the dilation
    $$D_{-\frac{5}{3}}$$
    A: $(\frac{3}{5}, \frac{3}{5})$ B: $(-\frac{3}{5}, -\frac{3}{5})$ C: (2,4) D: (0,9)

    E: (9,0) F: (−1,1)
12. Let t be the translation in Q X Q which has the following rule:
    $$(x,y) \qquad (x+\frac{5}{3}, y-\frac{2}{5}).$$
    (a) What is the rule for $t \circ t$?

    (b) What is the rule for t^3?

    (c) What is the rule for t^{-1} (the inverse of t)?

    (d) What is the rule for t^{-2}?

CHAPTER 15 INCIDENCE GEOMETRY

15.1 Preliminary Remarks.

In Chapter 13 we studied the properties of mass points. However, unlike the procedure in preceding chapters, we limited ourselves to properties which could be established through reasoning by deduction or deductive proof. It was found that if certain assumptions were made about the objects called mass points, many other properties were necessary consequences.

In this chapter we shall develop a similar deductive system. We will begin with some familiar words like *plane, line,* and *point* The axioms or assumptions about these objects will state some significant properties – already familiar from experience. Our task will be to show that many other properties of points, lines, and planes follow by deduction from the assumptions.

Since the axioms will be based on our experience with points, lines, and planes, whatever can be deduced from the axioms should also agree with experience. However, there may be properties of the plane which cannot be deduced from the limited number of axioms we will adopt. Although we will be dealing with objects called points, lines, and planes, we will not make use of any properties of these objects *except those stated precisely in the axioms.*

15.2 Axioms

We shall limit our entire discussion to the points and lines of a single plane which will be denoted by the letter "P". If you insist upon thinking of this plane as a flat surface like a floor, you may do so. However, the only real requirement imposed upon this plane is that it be a set of points. We will focus attention on certain subsets of the plane which have special properties.

Among these subsets are the lines (straight lines) of the plane. Again, if you insist upon thinking of a line as a taut wire, you may do so. We only insist that the line possess the properties which will be mentioned in the axioms.

The first axiom is given in two parts. In the first place, it requires that the plane contain at least two lines. A plane with only one line in it would hardly be much of a plane. The axiom also requires that each line contain at least two points. This certainly seems like a reasonable requirement. In fact, you probably feel that lines ought to have infinitely many points; we will not demand quite this much at present.

Axiom 1: (a) P contains at least two lines.

(b) Each line in P contains at least two points.

The second axiom also expresses a property that is reasonable to expect of lines and points. You will see that it plays an important part in our reasoning.

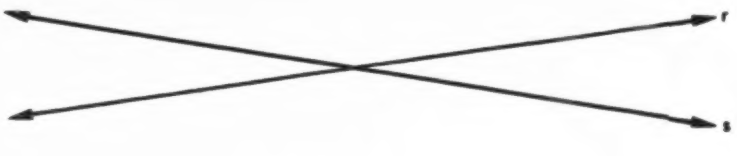
If some one were to ask you, "How many straight lines are there containing one particular point of a plane?" you would probably say, "As many as you want." But if you were asked, "How many straight lines are there containing two different points?" you would undoubtedly agree, "Just one." Certainly, whenever you draw a straight line through two points, A and B, you feel that there should be just one line, even though your drawing might not be accurate. At present we are not concerned about drawings but rather about ideas. The second axiom expresses a conviction about points and lines that you probably already have.

Axiom 2: For every two points in P there is one and only one line containing them.

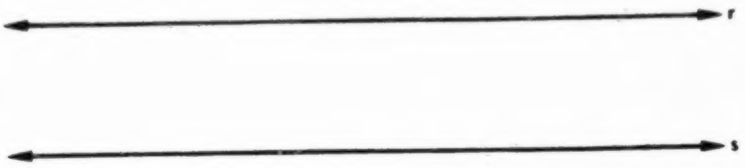
When we say "two points" we shall always mean two *distinct* points. If it should turn out that a single point happens to have two names, the conditions of Axiom 2 would not be satisfied, and we could not conclude that there is one and only one line containing this point. To allow for the possibility that a point or a line may have two names, we shall occasionally speak of a *pair* of points A and B. In such a case, A and B may (or may not) turn out to be the same point–depending upon other information we may have about A and B. Similarly, when we speak of a pair of lines c and d, these need not be distinct, but if we refer to the lines c and d, then it will be presumed that c and d are distinct lines.

Our third axiom deals with parallel lines. After we state it below, you will probably agree that it is a very reasonable requirement indeed. In fact, for two thousand years this axiom appeared so reasonable that many of the finest mathematicians thought that it was unnecessary to assume it. They felt that it should be possible to prove this particular property from the other axioms which had been adopted for Geometry. In other words, they thought that it ought to be a theorem rather than an additional axiom.

Before we state this axiom we should be clear about what we mean by "parallel lines". When we draw two lines, call them "r" and "s", on a sheet of paper, they may appear to intersect like this



or they may appear to not intersect like this

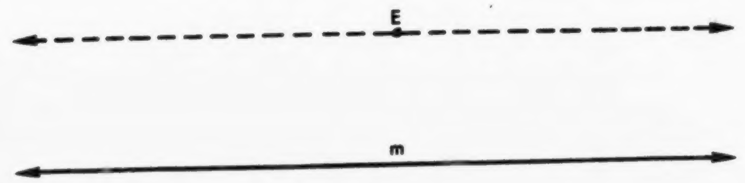


Of course, in the second case it is possible that r and s really do intersect. Perhaps if each line is extended sufficiently far beyond the confines of our sheet of paper, we would see that they actually meet. On the other hand, it might be difficult or perhaps impossible to decide this question in some cases. We certainly can conceive that lines r and s might never intersect; that is, $r \cap s = \phi$. In such a case we call lines r and s parallel. It is also convenient to consider r and s parallel even when $r = s$; that is, when r and s are the very same line. Accordingly, let us state the following definition.

Definition: Lines r and s in P are said to be *parallel* if $r = s$ or if $r \cap s = \phi$. When lines r and s are parallel, we express this fact by writing "$r \parallel s$".

The third axiom can now be stated

Axiom 3: For every line m and point E in the plane P, there is one and only one line containing E and parallel to m.

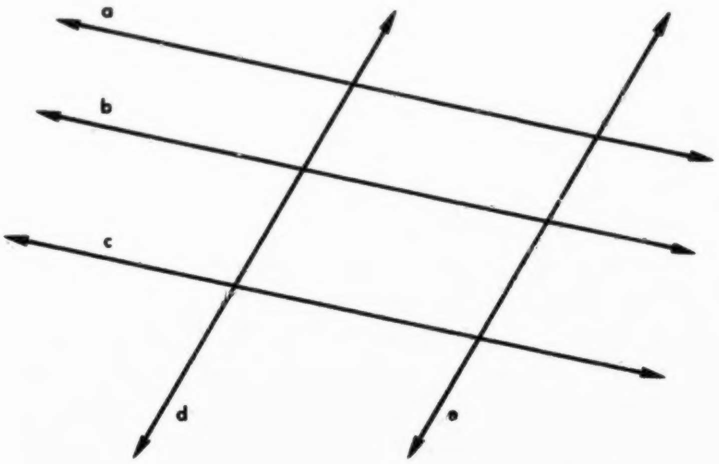


The need for such an axiom dealing with parallel lines was first recognized by Euclid who lived during the third century B. C. The axiom he adopted was the fifth in his list of axioms for geometry, and it corresponds closely to the one we have introduced here as our third axiom. The choice of this assumption was one of Euclid's great accomplishments for as we have noted, mathematicians for thousands of years after Euclid tried in vain to prove this reasonable property from the other axioms.

All these efforts were destined to failure because in the nineteenth century a number of great mathematicians (Gauss, Bolyai, Lobachevsky, and Riemann) showed that Euclid's fifth axiom did not follow from his other axioms. They proved this by creating perfectly good systems of geometry which did not have the property demanded by that axiom. Such systems are called non-Euclidean Geometries. If a system of geometry includes Euclid's fifth axiom, or any axiom equivalent to it, then that axiom is referred to as the Euclidean Axiom in the system.

15.3 Direction

What would you say if you were asked to describe the relationships among the lines of the following figure?



One possible answer would be, "a is parallel to b, b is parallel to c, a is parallel to c, d is parallel to e; and, a intersects d, a intersects e, b intersects d, b interesects e, c intersects d, and c intersects e."

Using the mathematical terminology of Chapter 8, the figure is a set of lines, $S = \{a, b, c, d, e\}$; there are two relations in S, "is parallel to" and "intersects". The relations can be indicated symbolically by "$a \parallel b$", "$c \parallel b$", "b I e", etc.

Another way of indicating the relations in S is to list the ordered pairs of lines meeting each condition (Remember that a relation is defined to be a subset of S X S). What pairs are needed to complete the listings begun here for "is parallel to" and "intersects"?

"is parallel to": (a,b), (b,c), (b,a), (a,a), . . .
"intersects": (a,d), (c,e), (e,c), . . .

You recall from Chapter 8 that certain relations in a set have interesting and useful properties. A relation in T is reflexive if and only if

$$tRt \text{ for each } t \text{ in } T;$$

it is symmetric if and only if for all t and s in T

$$tRs \text{ implies } sRt;$$

and it is transitive if and only if for all t, s, and q in T

$$tRs \textit{ and } sRq \text{ implies } tRq.$$

A relation which is reflexive, symmetric, and transitive is called an equivalence relation.

The first property to be deduced from our axioms is an important result concerning the relation "is parallel to".

Theorem 1: The relation "is parallel to" is an equivalence relation in the set of all lines in P.

Proof: All we need to do is check to see that the three conditions for an equivalence relation are satisfied by "$\parallel$".
(1) Is it true that for every line m in

P, m||m? If we look at the definition of parallel lines, we see that we agreed to consider every line as being parallel to itself. Therefore, the first condition for an equivalence relation is satisfied.

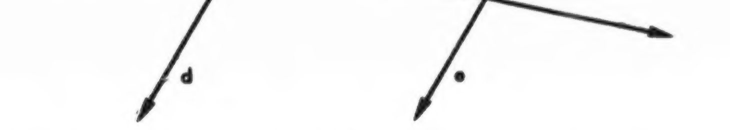
(2) If m and n are lines in P such that m||n, does it follow that n||m? Again we look at the definition of parallel lines. If m and n are the same line, there is nothing to prove. If n and m are distinct lines, then m and n have no points in common; if $m \cap n = \phi$, then $n \cap m = \phi$, so n||m.

(3) If m, n, and s are lines in P such that m||n and n||s, does it follow that m||s? Suppose it were not true that m||s. This would mean that m and s are distinct lines which have a point, A, in common. But then there would be two lines, m and s, containing A and parallel to n. This violates Axiom 3 which says that there can be only one line through A parallel to n. Therefore it follows that m and s cannot have a point in common or m||s. The third condition for an equivalence relation is satisfied.

Since "||" is reflexive, symmetric, and transitive, it is an equivalence relation.

The most significant property of an equivalence relation in a set is that it always partitions the set into disjoint subsets. The relation R puts elements a and b in the same subset or equivalence class if and only if aRb. How does the equivalence relation "is parallel to" partition the set of lines in P into disjoint subsets?

To get a picture of the way the equivalence classes are determined by "||", consider the figure shown earlier.



If lines which are related by "||" are put into the same class, the five lines pictured would be split into two classes: $S_1 = \{a, b, c\}$ and $S_2 = \{d, e\}$. In a similar manner "||" partitions the set of all lines in P into disjoint equivalence classes; each class consists of all the lines in P that are parallel to a given line.

We could say that all the lines in the same equivalence class run in the same direction or are in the same direction. In fact, we can refer to the equivalence classes as *directions* so that when lines are in the same equivalence class—that is, when they are parallel—they are in the same direction. Of course, if two lines are in different equivalence classes, they are not parallel and are not in the same direction.

It must be understood that when we use the word "direction" here, it has the same meaning as in the expression "the road runs in a north-south direction". The word "direction" does not have the same meaning as in the expression "the river flows in a southerly direction".

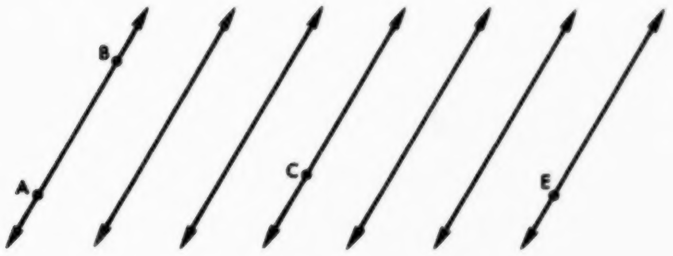
15.4 Exercises.

1. Is the relation "intersects" an equivalence relation in the set of all lines in P? Why or why not?
2. Which of the following determine equivalence relations for the specified sets?
   (a) "is the brother of" in the set of males.
   (b) "is the same age as" in the set of living people.
   (c) "is smaller than" in the set of students in your class.
   (d) "has the same number of pages as" in the set of books.
   (e) "is lighter than" in the set of students in your school.
   (f) "is the line reflection of (in a fixed line)" in the set of points in a plane.
   (g) "is perpendicular to" in the set of lines in a plane.
   (h) "has a point in common with" in the set of lines in a plane.
   (i) "is in the same grade as" in the set of students in your school.

   For each relation that actually is an equivalence relation, determine what kind of equivalence classes are formed.
3. Show that the relation "has the same author as" is an equivalence relation in the set of books in a bookstore. What kind of equivalence classes are determined by the relation?
4. Prove: If m is a line in P, then there is a point in P which is not in m. (Hint: Use both parts of Axiom 1 as well as Axiom 2.)
5. Prove: P has at least three lines. (Hint: Use problem 4 and Axiom 2.)

*6. Let D be a specified direction in P (equivalence class of parallel lines), and let R be the relation in P defined as follows:

Points A and B are in the relation R if and only if some line in direction D contains A and B.



In the sketch, $A \, R \, B$ but $C \not{R} E$.

Prove: 1) R is an equivalence relation in P.
2) The equivalence classes are the lines in the direction D.

*7. Prove: There are at least three directions in P. (Hint: Use problem 5)

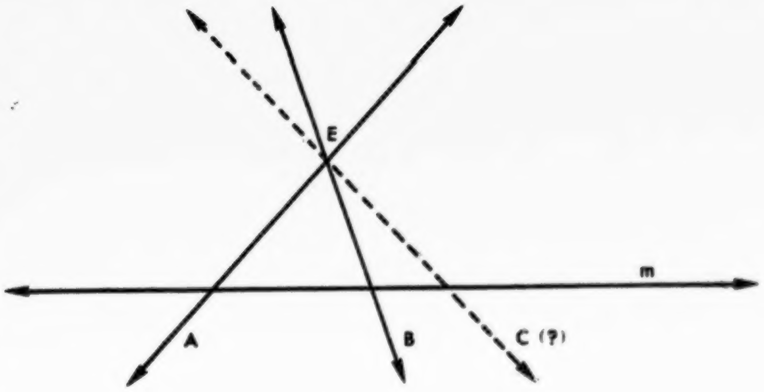
15.5 Some Consequences of the Axioms.

In Exercise 4 of the previous section you were asked to prove that there is a point not in a given line. Since we will use this result, a proof will now be given. You may want to check back and compare this proof with your own.

Theorem: If m is a line in P then there is a point in P which is not in m.

Proof: By Axiom Ia there is a line n distinct from the given line m; that is, $m \neq n$. By Axiom 1b there are distinct points A and B in n; that is, $A \neq B$. If both A and B were in m, then by Axiom 2 we would have $n = m$, which is not the case. Hence at least one of the points A or B is not in m.

Let us now consider line m, point E not in m, and all the lines containing E which intersect m. For example, $\overleftrightarrow{EA}$ and $\overleftrightarrow{EB}$ and perhaps another, $\overleftrightarrow{EC}$.



The next theorem simply says that there are "just as many" points in m as there are lines containing E which intersect m.

Theorem 3: In P, let m be any line and E any point not in m. Then there is a one-to-one correspondence between the points in m and the lines containing E which intersect m.

Proof: We must set up a correspondence between points and lines of P such that: (1) Each point in m corresponds to exactly one line which intersects m and contains E. (2) Each line which intersects m and contains E corresponds to exactly one point of m.

If A is any point in m, by Axiom 2 there is exactly one line which contains A and E. This line, $\overleftrightarrow{AE}$, intersects m. Let point A of m correspond to line $\overleftrightarrow{AE}$.

A
$\overleftrightarrow{AE}$

It remains to show that every line which intersects m and contains E is paired with exactly one point of m under the above correspondence. Assume n is such a line which intersects m in one point B. (Why can't m and n have two points in common?) Then B corresponds to $\overleftrightarrow{BE}$ under the above correspondence. But by Axiom 2 there is only one line containing B and E. Therefore, $\overleftrightarrow{BE} = n$ and n corresponds to B, a point of m.

15.6 Exercises.

1. Prove: There are at least four points in P. (Hint: Use Theorem 2 and Axiom 3.)

2. Prove: There are at least four lines in P. (Hint: Use Theorem 3 and Axiom 3.)

**3. Show that there need not be more than three directions in P and that each line in P need not contain more than 2 points. (Hint: To show this we need to construct a model of a "geometry" which has three directions and 2 points in each line. There will be objects *called* points and lines which have the properties specified in Axioms 1-3. However, these objects might be quite different from dots and straight lines on a paper. For instance, the "points" may be blobs of clay and the "lines" strips of wire.)

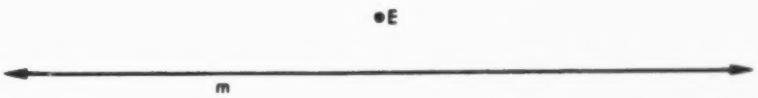
15.7 Parallel Projection.

Because we will need the result of Exercise 7 in Section 15.4, it will now be proved. You may want to compare your proof with the proof given below.

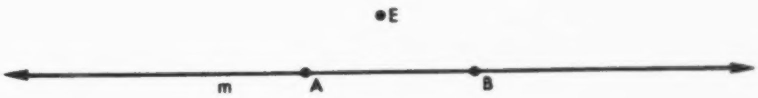
Theorem 4: There are at least three directions in P.

Proof: In Theorem 2 we proved that in P

there is a line m and a point E not in this line.

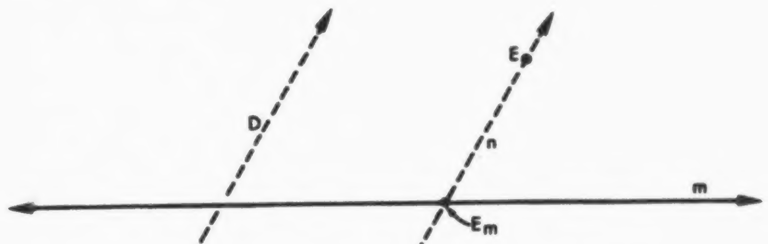


From Axiom 1 we know that m has at least two points, A and B.



Therefore, there are at least three distinct lines in P, $\overleftrightarrow{AB} = m$, $\overleftrightarrow{AE}$, and $\overleftrightarrow{BE}$. No two of these three lines can be parallel since each pair has a point in common: $\overleftrightarrow{AB} \cap \overleftrightarrow{AE} = A$, $\overleftrightarrow{AB} \cap \overleftrightarrow{BE} = B$, $\overleftrightarrow{AE} \cap \overleftrightarrow{BE} = E$. Therefore, the three lines determine three directions.

We shall now use the information that P has at least three directions. Let m be any line and D any direction not containing m. Let E be any point in P. From Axiom 3 we know that for every point E there is one and only one line, call it n, containing E which is in the direction D (i.e. n is parallel to a line in D).



Moreover, n cannot be parallel to m. If it was, then m would be in the direction of n which is D. We assumed that D was a direction *not* containing m. If n and m are in different directions, n and m are distinct lines that intersect in a point E_m. So for every line m and direction D not containing m we have a mapping that sends point E in the plane onto point E_m of line m. If we call this mapping "D_m", we have

$$D_m : E \longrightarrow E_m$$

Definition: We call the mapping D_m, that maps the points of P onto m, the *parallel projection* of P onto m in the direction D.

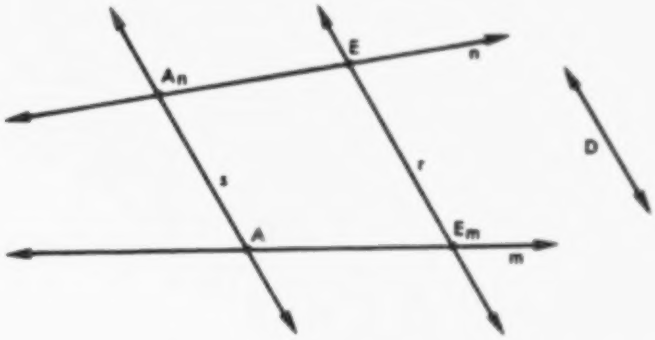
We now come to a very important theorem which makes use of almost all the information we have accumulated. It says that for any two lines in P, say m and n, there is a parallel projection that maps n one-to-one onto m.

Theorem 5: In P, let m and n be any lines and let D be any direction that contains neither m nor n. Then D_m is a parallel projection which maps n onto m. When the domain of D_m is restricted to n, D_m is one-to-one.

Proof: We must show two things.

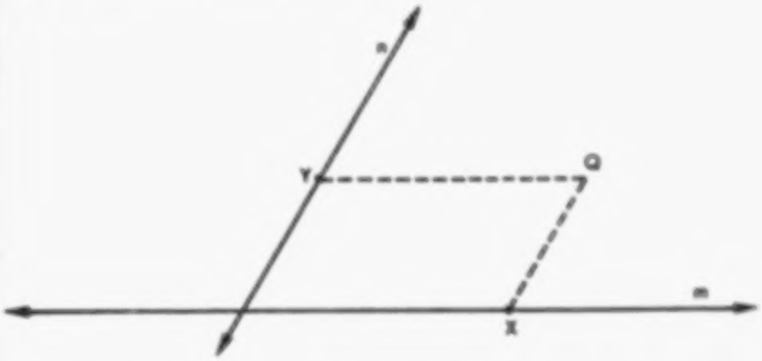
1) D_m maps each point of n onto some point of m.

2) Under the "restricted" mapping D_m, each point of m is the image of exactly one point in n.



Let us first show that D_m maps each point of n onto some point of m. Let E be any point of n. By Axiom 3 there is exactly one line in D, call it r, which contains E. We have selected direction D so that m and n are not in D. If follows then that $r \cap m \neq \phi$ and $r \neq m$. Hence, $r \cap m$ contains exactly one point, E_m. We have shown that D_m maps each point E of n onto some point, E_m, of m. To complete the proof we must show that when the domain of D_m is restricted to n, each point A of m is the image of exactly one point in n under this restricted mapping. Let s be a line in D which contains A. From Axiom 3 there is one and only one such line. As n is not in D, $s \cap n \neq \phi$ and $s \neq n$. If follows then that $s \cap n$ contains exactly one point, A_n. If there were another point in n which mapped onto A under D_m we would have two lines in D which contain A and this is impossible because the lines of D are parallel. We have completed the proof.

The notion of parallel projection constitutes the mathematical foundation on which one builds coordinate systems for locating points in a plane. One can choose any two lines m and n in different directions and use these lines as "coordinate axes".



It can now be shown that for each point Q in the plane, there is a unique ordered pair of points (X, Y) where X is in m and Y is in n. The points X and Y are determined by parallel projections onto m and n in the directions of n and m respectively. The pair of points (X,Y) then serve as coordinates of point Q.

15.8 Exercises.

1. What are the elements of:
   (a) plane P
   (b) a line
   (c) a direction
   (d) a relation
2. What do you mean by:
   (a) a line
   (b) the statement "lines r and s are parallel"
   (c) a direction
   (d) a relation
   (e) an equivalence relation
   (f) D_m
   (g) one-to-one correspondence
3. What are their own images under the mapping D_m?
4. What points have the same image, E_m, under the mapping D_m?
5. Is the composition $D_n \circ D_m$ a mapping of the "same kind" as D_m? (The domain of D_m is to be restricted to n)
6. Answer Sometimes, Always, or Never, whichever fits best.
   (a) Two points determine a direction.
   (b) Three points determine three lines.
   (c) If line n is in direction D and line m is not in D, then each point in n has the same image under D_m.
   (d) If two points A and B are such that their images under D_m are the same point, then $\overleftrightarrow{AB}$ is in direction D.
7. Prove: If r and s are any two lines in P, they have the same number of points. (Hint: Use the correspondence set up in the proof of Theorem 5)

15.9 Summary.

This chapter has dealt with a plane P which is simply a set of points with certain interesting subsets called lines. The lines were assumed to have the properties mentioned in the three axioms and from these properties we were able to deduce a number of further properties. It is important to note, however, that we were not able to deduce all the properties that we generally associate with lines and planes. For instance, Exercise 3, Section 15.6 showed that it is possible to have a "geometry" satisfying the axioms we selected in which each line has only two points.

The three axioms used are:
Axiom 1: (a) P contains at least two lines.
  (b) Each line in P contains at least two points.
Axiom 2: For every two points in P there is one and only one line containing them.
Axiom 3: In P, for every line m and every point E, there is one and only one line containing E and parallel to m.

Lines r and s are parallel if and only if $r = s$ or $r \cap s = \phi$. Using this definition we were able to prove that "is parallel to" is an equivalence relation in the set of lines in P. This relation partitions the set of lines in P into equivalence classes called directions, two lines being in the same direction if and only if they are parallel

The notion of a direction in P led to the following important consequences of the axioms:

(a) There are at least three directions in P.
(b) For a fixed direction the following relation R is an equivalence relation on P: For points A and B, A R B if and only if there is a line in D which contains A and B.
(c) To every direction D and line m not in D there is a parallel projection, D_m, which maps all the points of P onto m.
(d) For every two lines m and n in P there is a parallel projection that maps n onto m and is one-to-one.

15.10 Review Exercises.

1. If m is any line in P, prove that there are at least two points not in m.
2. If m is any line in P, prove that there are at least two directions not containing m.
3. If lines m, n, and s are distinct lines in P such that $m \| n$ and $n \| s$, prove that $m \| s$.
4. If lines m, n, and s are distinct lines in P such that $m \| n$, and s intersects m, then s intersects n.

*5. Prove that if D_1 and D_2 are two directions then there is a one-to-one correspondence between all the lines of D_1 and all the lines of D_2.